Abstract of thesis entitled

ON k-TUPLES OF ALMOST PRIMES

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Call an integer an E_j-number if it is a product of j distinct primes. It is proved that for infinitely many n,

(I) n is an E_2-number, at the same time $n+2$ is either an E_2 or an E_1-number.

Also, by calling an integer a P_j-number if it is a product of at most j primes, it is proved that for infinitely many n,

(II) (a) n is a prime, at the same time $n+2$ is a P_3-number.
(b) n is a P_5-number, at the same time $n+2$ is a P_4, $n+6$ is a P_3, and $n+8$ is a P_2-number.

These results can be well extended to any admissible 2 or 4-tuples. An admissible k-tuple is a set of k linear functions $\{a_1n+b_1, \dots, a_kn+b_k\}$ satisfying certain arithmetic conditions. Let " $\lceil k : r_k\, , R_k \rfloor$" denote the following statement:

> Given any admissible k-tuple $\{a_1n+b_1, \dots, a_kn+b_k\}$, there exist infinitely many n such that the product $(a_1n+b_1) \times \cdots \times (a_kn+b_k)$ is a P_{r_k}-number, at the same time each a_in+b_i is a P_{R_k}-number.

Then (III) $\lceil 3 : 8\, , 3 \rfloor$, $\lceil 4 : 12\, , 4 \rfloor$, $\lceil 4 : 11\, , 5 \rfloor$, $\lceil 5 : 15\, , 5 \rfloor$ are proved as well.

The proofs are developed from the approach of using Λ^2 sieve to study admissible k-tuples. By using Selberg's idea, a k-variable Λ^2 sieve, $\lambda_d \to \lambda_{\delta_1,\dots,\delta_k}$, is introduced. The truncation of the sieve can therefore be better controlled, which helps in the proof of (I). By evaluating the asymptotic behaviour of sifted E_j-numbers, the following expression can be considered:

$$\sum_{N<n\le 2N} \left[\sum_{i=1}^{k} \sum_{j=1}^{L} \alpha_{ij}\gamma_j(a_in+b_i) - B \right] \Lambda^2,$$

where $\gamma_j : \mathbb{N} \to \{0,1\}$ are functions supported on E_j-numbers, and α_{ij} are to be determined. (II) and (III) are proved in this way.

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Declaration

I declare that this thesis represents my own work, except where due acknowledgement is made, and that it has not been previously included in a thesis, dissertation or report submitted to this University or to any other institution for a degree, diploma or other qualifications.

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Contents

Turn to the left, then bear right, and you will touch the intangible, you will reach the inaccessibly remote tracts of which one never knows anything on this earth except the direction, except . . . the ‘way.’

– Marcel Proust, *In Search of Lost Time: Time Regained*

Chapter 1

Introduction

1.1 Prolegomena

"There are infinitely many $n \in \mathbb{N}$ such that both n and $n+2$ are primes."

This is the famous twin prime conjecture. Although the statement is well supported by heuristic arguments and computational results, to date it has not been proved or disproved. We have a result towards its proof, which is due to Chen. He showed that there are infinitely many prime p such that $p+2$ is either a prime or a product of two primes, c.f. [8, chapter 11] for example. Several decades have passed since Chen's theorem was published, but we are still unable to prove the conjecture.

The story does not end here. One should be inspired by this conjecture and consider the following generalized problem:

"Given a set of $k \geq 2$ linear functions $\{a_1n+b_1, \ldots, a_kn+b_k\}$, where $a_i, b_i \in \mathbb{Z}$ and $a_i > 0$, do there exist infinitely many $n \in \mathbb{N}$ such that all a_in+b_i are primes simultaneously?"

We immediately find that the answer is *no* in some cases, say $\{n, n+1\}$ for example, because for each $n \in \mathbb{N}$, one of the pair can be divided by 2. Such trivial cases should first be excluded from our consideration.

Definition 1.1.1

A k-tuple $\mathfrak{L} = \{a_1n+b_1, \ldots, a_kn+b_k\}$, where $a_i, b_i \in \mathbb{Z}$, $a_i > 0$, and $a_ib_j - a_jb_i \neq 0$ for all $i \neq j$ [1]*, is called* ***admissible****, if by defining for each prime p*

$$\nu_p(\mathfrak{L}) := \#\{1 \leq n \leq p : p \text{ divides } \textstyle\prod_{i=1}^{k}(a_in+b_i)\},$$

we always have $\nu_p(\mathfrak{L}) < p$.

[1] This is to ensure that the a_in+b_i are distinct with respect to constant multiplications.

By restricting our attention to admissible k-tuples, the aforementioned trivial situation is exactly eliminated. We note that given k, we can always find an admissible k-tuple, for one can consider $\mathfrak{L}$ given by $\{n+p_1,\ldots,n+p_k\}$, where $p_1,\ldots,p_k$ are distinct primes greater than k.

It is time to state the prime k-tuple conjecture formally:

Conjecture 1.1.2 (Prime k-tuple conjecture, qualitative form)

Let $\mathfrak{L}$ be an admissible k-tuple, $k \geq 2$. Then there are infinitely many $n \in \mathbb{N}$ such that all the $a_i n+b_i$ involved are primes simultaneously.

As the name suggests, there is a quantitative form. This can be found in, e.g. Hardy and Littlewood[9, Theorem X1], where they discovered it through a heuristic argument with the circle method (for the case where all $a_i = 1$). We shall introduce the conjecture by a rather straightforward approach, which makes use of a probabilistic model suggested by Soundararajan[17].

Let N be a large real number. Then for any $n \in \mathbb{N}$ such that its order is comparable to N, we assume that the probability of n being a prime number is $\frac{1}{\log N}$, as the prime number theorem suggests. In this way, the size of the set

$$\{N<n\leq 2N : a_i n+b_i \text{ is a prime for all } i=1,\ldots,k\}$$

is of order $\frac{N}{\log^k N}$. We would like to add a correction factor, because the $a_i n+b_i$ are not really independent. For example, in the case $\mathfrak{L} = \{n, n+2\}$, if one of the pair can be divided by 2, so is the other.

Given a prime p, the probability that p does not divide k randomly chosen integers is $(1-\frac{1}{p})^k$, while the probability that p does not divide $\prod_{i=1}^k (a_i n+b_i)$ is $(1-\frac{\nu_p(\mathfrak{L})}{p})$. This suggests that the correction factor should be

Definition 1.1.3

$$\mathfrak{S} = \mathfrak{S}(\mathfrak{L}) := \prod_p \left(1-\frac{\nu_p(\mathfrak{L})}{p}\right)\left(1-\frac{1}{p}\right)^{-k}.$$

So[2] we arrive at the following conjecture:

Conjecture 1.1.4 (Prime k-tuple conjecture, quantitative form)

Let $\mathfrak{L}$ be an admissible k-tuple, $k \geq 2$. Then for any large N, we have

$$\#\{N<n\leq 2N : a_i n+b_i \textit{ is a prime for all } i=1,\ldots,k\} \sim \mathfrak{S}\frac{N}{\log^k N}.$$

[2]Observe that $\mathfrak{S} < \infty$, because the requirement $a_i b_j - a_j b_i \neq 0$ implies $\nu_p(\mathfrak{L}) = k$ for all large prime p.

We are far away from proving the prime k-tuple conjecture, because even the single case $k = 2$ still defeats us. Nevertheless, as Chen's theorem suggests, inferior results are possible if one relaxes the requirement from being primes to being almost primes. Almost primes are those integers with few numbers of prime factors. For convenience of later discussion, we first make the following definition:

Definition 1.1.5

We write "$\lceil k : r_k \,, R_k \rfloor$" to denote the following statement:

> *Given any admissible k-tuple $\{a_1 n + b_1, \ldots, a_k n + b_k\}$, there exist infinitely many n such that the product $(a_1 n + b_1) \times \cdots \times (a_k n + b_k)$ has at most r_k prime factors, and at the same time each $a_i n + b_i$ has at most R_k prime factors, all counting multiplicity.*[3]

As an example, Chen's theorem is the result $\lceil 2 : 3 \,, 2 \rfloor$.

This opens a new path of research. Namely, although we are unable to reach $\lceil k : k \,, 1 \rfloor$ in the meantime, we can try to prove $\lceil k : r_k \,, R_k \rfloor$ with r_k and R_k as small as possible. To this end, sieve method enters naturally, for it can be used to sift almost primes from integers. To date, the best results for the case of general k are produced by the Λ^2 sieve method, which we shall describe shortly.

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The Λ^2 sieve method has been invented and examined by Selberg since 1940s. One can find in his work [16, Section 23] the following strategy to tackle the twin-prime conjecture:

Let τ be the divisor function, and let $\lambda = \lambda_d$ be a real-valued function in d. Suppose one succeeds in showing that for some positive number B, we have

$$\sum_{N < n \leq 2N} \Big[B - \tau(n) - \tau(n+2) \Big] \times \left(\sum_{d \mid n(n+2)} \lambda_d \right)^2 > 0$$

for all large N. Then the conclusion $B > \tau(n) + \tau(n+2)$ for infinitely many n follows immediately, because $\left(\sum_{d \mid n(n+2)} \lambda_d \right)^2$ is always nonnegative. Upper bounds for r_2 and R_2 are thus obtained from the value of B.

[3] Observe that we can have $R_k \leq r_k - (k-1)$ and $r_k \leq k R_k$.

Observe that the smaller the B, the better is the outcome. Since B is determined by the ratio of the following two expressions:

$$\begin{aligned} S_1 &:= \sum_{N<n\leq 2N} 1 \left(\sum_{d|n(n+2)} \lambda_d \right)^2, \\ S_2 &:= \sum_{N<n\leq 2N} [\tau(n)+\tau(n+2)] \left(\sum_{d|n(n+2)} \lambda_d \right)^2, \end{aligned}$$

therefore our task is to choose λ suitably so that $\frac{S_2}{S_1}$ is minimized. Note that $\sum_{N<n\leq 2N} 1$ has order N, while $\sum_{N<n\leq 2N}[\tau(n)+\tau(n+2)]$ has order $N \log N$. In order to bring them into the same order of magnitude, the auxiliary $\Lambda^2 = \left(\sum_{d|n(n+2)} \lambda_d\right)^2$ should be able to sift out those n with $\tau(n)+\tau(n+2)$ large from the sum, or equivalently, the weight of Λ^2 should be concentrated on those n such that $\tau(n)+\tau(n+2)$ is small.

Therefore, the Λ^2 is called a sieve, and we are in the position of finding a good one to minimize $\frac{S_2}{S_1}$. Selberg considered this as an intractable problem, and his choice of λ was just to minimize S_2. He eventually obtained the result $\lceil 2:5,3 \rfloor$.

We remark that Selberg chose his λ by a diagonalization argument, which is a common practice when using the Λ^2 sieve method.[4] However, when the aforementioned methodology becomes well adopted to the prime k-tuple problem, it appears that diagonalization cannot take its place in optimization. As a result, what the square in Λ^2 brings to us is mainly nonnegativity.

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A natural way to extend Selberg's idea is to generalize S_1, S_2 to the case of admissible k-tuple and find a good Λ to reduce the corresponding ratio. This is the work of Heath-Brown [10]. The sieve he chose for

$$\Lambda^2 = \left(\sum_{d|\prod_{i=1}^{k}(a_i n+b_i)} \lambda_d \right)^2$$

can be defined by

$$\lambda_d := \begin{cases} \mu(d)\left(\frac{\log z/d}{\log z}\right)^{k+1} & \text{if } d \leq z, \\ 0 & \text{otherwise}, \end{cases} \tag{1.1}$$

where μ is the Möbius function.

[4] One may refer to [16, Section 7] for the approach and [15, Theorem 1 and Lemma 7] for an application to the Riemann Hypothesis.

We note the following points about this λ:

1. Let m have the prime factorization $p_1^{w_1}\cdots p_r^{w_r}$. We have

$$\sum_{d|m}\mu(d)\left(\log\frac{z}{d}\right)^D=\sum_{i_1=0}^{1}\cdots\sum_{i_r=0}^{1}(-1)^{i_1+\cdots+i_r}(\log z-\log p_1^{i_1}-\cdots-\log p_r^{i_r})^D.$$

   Using the multinomial expansion and changing the order of summation, the right hand side is

$$=\sum_{\substack{j_1,\ldots,j_{r+1}\\ j_1+\cdots+j_{r+1}=D}}\sum_{i_1=0}^{1}\cdots\sum_{i_r=0}^{1}\frac{(-1)^{i_1+\cdots+i_r}D!\log^{j_{r+1}}z}{j_1!\cdots j_{r+1}!}\prod_{\ell=1}^{r}(-\log p_\ell^{i_\ell})^{j_\ell},$$

   where we take $(-\log p_\ell^{i_\ell})^{j_\ell}=1$ when $i_\ell, j_\ell=0$.

   Observe that for $1\le\ell\le r$, the term corresponds to $j_\ell=0$ is $1+(-1)=0$. Hence, if $r>D$, we have $\sum_{d|m}\mu(d)(\log z/d)^D=0$. This demonstrates the sifting property of Λ in case no truncation ($d\le z$) is imposed.[5]

2. The a_in+b_i are related by congruence conditions. As an example, let us consider $\mathfrak{L}=\{n,n+2,n+6\}$. Clearly, if $\delta|n$ then $(n+2)\equiv 2(\mathrm{mod}\ \delta)$, and if $\delta|(n+6)$ then $(n+2)\equiv -4(\mathrm{mod}\ \delta)$. Hence, when evaluating

   (1.2)
$$\sum_{N<n\le 2N}\tau(n+2)\Lambda^2 = \sum_{N<n\le 2N}\tau(n+2)\left(\sum_{d|n(n+2)(n+6)}\lambda_d\right)\left(\sum_{\widetilde{d}|n(n+2)(n+6)}\lambda_{\widetilde{d}}\right),$$

   by letting each $d|n(n+2)(n+6)$ be $d=\delta_1\delta_2\delta_3$, $\delta_1|n$, $\delta_2|(n+2)$, $\delta_3|(n+6)$, we find that we have to deal with a sum of the form

$$\sum_{\substack{m\le x\\ m\equiv a(\mathrm{mod}\ q)}}\tau(m), \tag{1.3}$$

   where the size of q is determined by δ_1,$\widetilde{\delta}_1$,δ_3,$\widetilde{\delta}_3$, while the size of x is determined by N and δ_2, $\widetilde{\delta}_2$. The sum (1.3) can be handled efficiently only when q does not exceed certain size with respect to x.[6] The truncation $d\le z$ is imposed for this purpose.

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[5] The reason why $D=k+1$ instead of $D=k$ was used in (1.1) is a technical one: for otherwise S_2 fails to have an asymptotic behaviour.

[6] To date this is allowed to be about $x^{2/3}$. The exponent $\frac{2}{3}$ is called the "level of distribution".

By this choice of λ, Heath-Brown not only recovered Selberg's $\lceil 2 : 5\,,3 \rfloor$ result, but he also obtained general $\lceil k : r_k\,, R_k \rfloor$ results. We do not report them here, because by a neat refinement of the λ, Ho and Tsang [11] was able to give improved result.

The sieve candidate Ho and Tsang used was

$$\lambda_d = \mu(d) \sum_{\ell=k+1}^{L} \alpha_\ell \left(\frac{\log z/d}{\log z} \right)^{\ell},$$

where the α_ℓ's were determined by optimizing the ratio $\dfrac{S_2}{S_1}$. This brings flexibility to the process of optimization, thus resulting in improvements. In terms of R_k, their results are the best for $k \geq 3$.

Their results are as follows:

Ho and Tsang [11]
$\lceil 2 : 5\,,3 \rfloor$
$\lceil 3 : 8\,,4 \rfloor$
$\lceil 4 : 12\,,4 \rfloor$
$\lceil 5 : 16\,,5 \rfloor$
$\lceil 6 : 20\,,5 \rfloor$
$\lceil 7 : 24\,,6 \rfloor$
$\lceil 8 : 28\,,6 \rfloor$
$\lceil 9 : 33\,,6 \rfloor$
$\lceil 10 : 38\,,7 \rfloor$

* * * * * * * * *

Apart from optimizing the sieve, the next thing is to examine the complementary weight. By a complementary weight we mean a function f such that $f(n)$ can provide information about the number of prime factors of n. What we have seen so far is that $f(n) = \tau(n)$. Let's try another one, say $f(n) = \varpi(n)$, where ϖ is the characteristic function of primes. Suppose temporarily $a_1 = a_2 = \cdots = a_k$, and that we are able to show that for some k,

$$\sum_{N<n\leq 2N} \left[\sum_{i=1}^{k} \varpi(a_i n + b_i) - 1 \right] \Lambda^2 > 0$$

for all large N. Then this would imply bounded gaps between successive primes exist[7].

[7] i.e. $\liminf_{n\to\infty}(p_{n+1} - p_n) < \infty$, where p_n denotes the n^{th} prime.

One can expect that this approach, if works, requires k to be rather large. After all, by the current technology, sieve method is not that efficient at purely picking primes. Therefore, to keep k small, we should involve almost primes as well, as illustrated by Chen's Theorem. In this direction, we have the work by Goldston, Graham, Pintz and Yıldırım[5]. We first make a definition before giving the description.

Definition 1.1.6

A positive integer is called an ***E_j-number*** *if it is a product of exactly j distinct primes.*

For instance, 6 is an E_2-number and 42 is an E_3-number, but 9 and 50 are not E_j-numbers for any j.

Now we can account for their work in this way:

1. Let $\gamma_2 : \mathbb{N} \to \{0,1\}$ be a function supported on E_2-numbers. i.e. $\gamma_2(n)=1$ only if n is an E_2-number.

2. By showing that

$$\sum_{N<n\leq 2N} \begin{bmatrix} \gamma_2(a_1n+b_1) & + & \gamma_2(a_2n+b_2) & \\ + & & + & -1 \\ \varpi(a_1n+b_1) & + & \varpi(a_2n+b_2) & \end{bmatrix} \Lambda^2 > 0 \tag{1.4}$$

   for all large N, they obtained the result $\lceil 2 : 4\,, 2 \rfloor$.

3. By showing that

$$\sum_{N<n\leq 2N} [\gamma_2(a_1n+b_1) + \gamma_2(a_2n+b_2) + \gamma_2(a_3n+b_3) - 1]\,\Lambda^2 > 0$$

   for all large N, they found that for any admissible 3-tuple, infinitely many often two of the a_in+b_i involved are E_2-numbers simultaneously.

In fact, since $\{n, n+2, n+6\}$ is an admissible 3-tuple, we can conclude that the gaps between consecutive E_2-numbers can be ≤ 6 infinitely many often. One then wonders whether the following conjecture is true:

Conjecture 1.1.7 (Twin E_2 Conjecture)

There exist infinitely many n such that both n and $n+2$ are E_2-numbers.[8]

[8]Examples include $\{33, 35\}$, $\{779, 781\}$. We consider $\{n, n+2\}$ rather than $\{n, n+1\}$ because the latter is a consequence of the following conjecture: there are infinitely many prime p such that $2p+1$ is an E_2-number. See [6, Section 1].

We shall talk more about the problems of gaps between primes in Chapter 4. We stop the discussion here by noting that, as mentioned earlier for the case $f(n) = \tau(n)$, one needs information about sums of $\varpi(n)$ and $\gamma_2(n)$ over arithmetic progressions. The first is provided by Bombieri-Vinogradov theorem, and the second is provided by the work of Motohashi[13]. The size allowed for the moduli is about $x^{1/2}$ in both cases.

* * * * * * * * *

It is Maynard [12] who employed the idea of counting E_j-numbers in the prime k-tuple problem. The complementary weight he used can be defined by $f(n) := j-2$ if n is an E_j-number. Plainly, the expression under consideration is

$$\sum_{N<n\le 2N}\left[B-\sum_{i=1}^{k} f(a_in+b_i)\right]\Lambda^2.$$

To evaluate $\sum_{N<n\le 2N} f(a_in+b_i)\Lambda^2$, observe that if the sum is restricted to square free integers, this is

$$\begin{aligned}
&\sim \sum_{N<n\le 2N}\sum_{p|a_in+b_i}\left[1-\frac{2\log p}{\log N}\right]\Lambda^2\\
&\sim \sum_{N<n\le 2N}\sum_{\substack{p|a_in+b_i\\ p\le\sqrt{N}}}\left[1-\frac{2\log p}{\log N}\right]\Lambda^2+\sum_{N<n\le 2N}\sum_{j=1}^{\infty}\left[w(a_in+b_i)\varpi_j(a_in+b_i)\right]\Lambda^2,
\end{aligned}$$

where ϖ_j is the characteristic function of E_j-numbers, and $w(m)$ gives the weight

$$\left(1-\frac{2\log p_{\max}}{\log N}\right)$$

if the largest prime factor $p_{\max}$ of m is $> N^{\frac{1}{2}}$, and $w(m) = 0$ otherwise. Since every term in the second double sum is nonpositive, so by truncating the sum over j to $1 \le j \le L$, say, and using Motohashi result, we can get a computable upper bound for the sum above.

In terms of r_k, to date this method produces the best results for $k \ge 4$. They are recorded and compared with the results of Ho and Tsang as follows.

Ho and Tsang [11]	Maynard [12]
$\lceil 2:5,3\rfloor$	$\lceil 2:5,4\rfloor$
$\lceil 3:8,4\rfloor$	$\lceil 3:8,6\rfloor$
$\lceil 4:12,4\rfloor$	$\lceil 4:11,8\rfloor$
$\lceil 5:16,5\rfloor$	$\lceil 5:15,11\rfloor$
$\lceil 6:20,5\rfloor$	$\lceil 6:18,13\rfloor$
$\lceil 7:24,6\rfloor$	$\lceil 7:22,16\rfloor$
$\lceil 8:28,6\rfloor$	$\lceil 8:26,19\rfloor$
$\lceil 9:33,6\rfloor$	$\lceil 9:30,22\rfloor$
$\lceil 10:38,7\rfloor$	$\lceil 10:34,25\rfloor$

Maynard attributed the improvement in r_k to the following observation: although we have stronger level-of-distribution result for $\tau(n)$ in arithmetic progressions (i.e. $\frac{2}{3} > \frac{1}{2}$), the divisor function assigns much larger weight to those n with many prime factors than f, rendering the siftings less noticeable. But then we are in a catch-22 situation, because it is this fact that enables us to exclude the possibility that some $a_i n+b_i$ contains many prime factors. As a result, in terms of R_k, Maynard's results are not as impressive as that of Ho and Tsang.

1.2 Exploration in this thesis

1. *On the sieve.*

   As mentioned earlier in the previous section, one is required to impose a truncation on λ_d in order to handle a sum over arithmetic progressions. The more the truncation, that is, the fewer the nonzero λ_d available, the weaker would be the sifting effect.

   We find that there is always a bit overflow of the truncation imposed in previous studies. Again consider the previous example $\mathfrak{L} = \{n, n+2, n+6\}$ and the sums (1.2) and (1.3). It has been observed that the role of δ_2 is different from that of δ_1, δ_3. As Selberg [16, Section 23] suggested, if we consider a more flexible multivariable Λ^2 sieve,

$$\Lambda^2 = \left(\sum_{\substack{\delta_1 | n \\ \delta_2 | n+2 \\ \delta_3 | n+6}} \lambda_{\delta_1,\delta_2,\delta_3} \right)^2 ,$$

   then the asymmetry can be handled more properly, resulting in a relaxation of the truncation on λ. Previous studies do not do so, perhaps due to simplicity and the fact that the difference is small when k is large. Despite the complexity, improved results should be possible by this new approach, at least when k is small.

   Exploring this idea is the main theme of Chapter 3, in which we shall obtain the following result:

   <u>Theorem 1.2.1</u>

   For any admissible 2-tuple $\{a_1n+b_1, a_2n+b_2\}$, there exist infinitely many n such that a_1n+b_1 is an E_2-number, and at the same time a_2n+b_2 is either an E_2-number or a prime.

Although this is inferior to Chen's theorem when one considers the twin prime conjecture, this can nonetheless be regarded as an analog of Chen's result to the twin E_2 conjecture.

To achieve this result, we shall demonstrate that by our new sieve, we can show

$$\sum_{N<n\leq 2N}\left[\gamma_2(a_1n+b_1)\begin{array}{c}\gamma_2(a_2n+b_2)\\+\\\varpi(a_2n+b_2)\end{array}-1\right]\Lambda_{\text{new}}^2>0$$

for all large N. The result then follows immediately.

We have seen similar expression in the previous section in the discussion of the work in [5], namely (1.4), in which there are two ϖ functions instead of one. We note that it is our new sieve that allows us to drop one ϖ while still keeping the sum >0 for all large N. The previous sieves fail to do so.

2. *On the weight.*

In [8, Chapter 10], Halberstam and Richert described the following general sieve procedure: Let $f(n)$ be an arithmetic function which increases with $\omega(n)$, the number of prime factors n possesses. If for some choice of B and Λ^2 one has

$$\sum_{n\in S}[B-f(n)]\Lambda^2>0,$$

then there exists $n\in S$ such that $f(n)<B$. In view of the nature of f, we then have an upper bound of $\omega(n)$. Moreover, at the end of the description, they commented that this method is not readily applicable, due to our lack of knowledge of the distribution of a general arithmetic function f over arithmetic progressions. Finally, they hope for finding a Bombieri-Vinogradov-type theorem for such general f.

Perhaps here we can give a response to their comment, at least for the case of the prime k-tuple problem that we are considering. First of all, we assume that until the end of this section, all integers under consideration are square free. Then for any arithmetic function f such that $f(n)$ only depends on $\omega(n)$, we have

$$f(m)=\sum_{j=1}^{\infty}\varpi_j(m)f(p_1\cdots p_j),$$

where p_i denotes the i^{th} prime. For instance, we have

$$\tau(m)=\sum_{j=1}^{\infty}\varpi_j(m)2^j\qquad\text{and}\qquad\omega(m)=\sum_{j=1}^{\infty}\varpi_j(m)j.$$

If f is a finite linear combination of ϖ_j, then by the result of Motohashi[13], we have such a corresponding Bombieri-Vinogradov-type theorem, with the level of distribution to be $\frac{1}{2}$. We now demonstrate that it is the nature of this sieve procedure that allows us to write f as a finite linear combination of ϖ_j. As an example, consider the twin prime problem. Assume that for a fixed choice of Λ^2, we have

$$\sum_{N<n\leq 2N} [20 - \tau(n) - \tau(n+2)]\Lambda^2 > 0$$

for all large N, and that 20 is the smallest possible integer for this to be true. So we have the result $\lceil 2 : 6\,, 4 \rfloor$. Then when we replace τ by

$$\tau'(m) := 2\varpi_1(m) + 4\varpi_2(m) + 8\varpi_3(m) + 16\varpi_4(m) + \sum_{j=5}^{\infty} 20\varpi_j(m),$$

which satisfies $\tau'(m) \leq \tau(m)$, a fortiori we have

$$\sum_{N<n\leq 2N} [20 - \tau'(n) - \tau'(n+2)]\Lambda^2 > 0. \tag{1.5}$$

But in this way,

(i) The result $\lceil 2 : 6\,, 4 \rfloor$ still holds.

(ii) It is possible to choose the integer 20 in (1.5) to be smaller, while the expression is still positive for all large N. Thus, we may get an improved $\lceil 2 : r_2\,, R_2 \rfloor$ result.

(iii) τ' is a finite linear combination of ϖ_j:

$$\tau'(m) = \sum_{j=1}^{4} 2^j \varpi_j(m) + (20)\left(1 - \sum_{j=1}^{4} \varpi_j(m)\right),$$

and so

$$20 - \tau'(n) - \tau'(n+2) = \sum_{j=1}^{4} \alpha_j \varpi_j(n) + \sum_{j=1}^{4} \alpha_j \varpi_j(n+2) - 20$$

for some $\alpha_j > 0$.

This observation suggests that, when we are using this sieve procedure, if we are prepared to accept $\frac{1}{2}$ to be the level of distribution, and that $f(n)$ only depends on and increases with $\omega(n)$, then we can obtain the best results by cutting off f so that $f(n)$ is a constant for all $\omega(n) > L$, say, and the expression we need to consider is

$$\sum_{N<n\leq 2N} \left[\sum_{i=1}^{k} \sum_{j=1}^{L} \alpha_{ij}\varpi_j(a_i n + b_i) - B\right]\Lambda^2, \tag{1.6}$$

where α_{ij} are to be determined.

This is the main theme of Chapter 4. We shall in particular obtain the following result:

Theorem 1.2.2

The following statements are true.

(a) *For any admissible 2-tuple $\{a_1n+b_1, a_2n+b_2\}$, there exist infinitely many n such that a_1n+b_1 is a prime number, and at the same time a_2n+b_2 is either a prime, or an E_2, or an E_3-number.*

(b) $\lceil 3:8\,,3\rfloor$.

(c) $\lceil 4:12\,,4\rfloor$.

(d) $\lceil 4:11\,,5\rfloor$.

(e) $\lceil 5:15\,,5\rfloor$.

Although Result(a) is again weaker than Chen's one, it should be worth mentioning, not only for the sake of completeness, but also due to the fact that it is obtained through a non-linear sieve.

When we use the following table to compare our theorem with the previous results,

Ho and Tsang [11]	Results in Theorem 1.2.2	Maynard [12]
$\lceil 2:5\,,3\rfloor$	Result(a)	$\lceil 2:5\,,4\rfloor$
$\lceil 3:8\,,4\rfloor$	$\lceil 3:8\,,3\rfloor$	$\lceil 3:8\,,6\rfloor$
$\lceil 4:12\,,4\rfloor$	$\lceil 4:12\,,4\rfloor/\lceil 4:11\,,5\rfloor$	$\lceil 4:11\,,8\rfloor$
$\lceil 5:16\,,5\rfloor$	$\lceil 5:15\,,5\rfloor$	$\lceil 5:15\,,11\rfloor$

we see that we can always reach some kind of improvement. It will also be clear that our result can be extended beyond $k=5$.

As revealed by Theorem 1.2.2(a), we can have other interesting results besides that of the form $\lceil k:r_k\,,R_k\rfloor$. As an example, we shall obtain:

Theorem 1.2.3

For any admissible 4-tuple $\{a_1n+b_1, a_2n+b_2, a_3n+b_3, a_4n+b_4\}$, there exist infinitely many n such that a_1n+b_1 is a P_5-number, a_2n+b_2 is a P_4-number, a_3n+b_3 is a P_3-number, and a_4n+b_4 is a P_2-number.[9]

In particular, this theorem can apply to $\{10n+1, 10n+3, 10n+7, 10n+9\}$, which is called a prime decade when all the members are primes.

This section is ended by remarking that we succeed in proving Theorem 1.2.2 and Theorem 1.2.3 because the weights assigned to the E_j-numbers, namely the α_{ij} in (1.6), can be chosen freely. This allows us to construct sophisticated weights in application.

[9]By a P_j-number we mean an integer with at most j prime factors.

1.3 Notations and Conventions

Throughout the thesis, we shall make use of the following notations and conventions:

With or without subscripts, the symbol p will be used to denote prime numbers, s and ξ to denote complex variables with σ=Re(s) and t=Im(s).

For $r \in \mathbb{R}$, we write (r) to denote the vertical line in $\mathbb{C}$ that passes through r. For $a > 0$, we let $\mathbb{H}_{-a} := \{s \in \mathbb{C} : \sigma > -a\}$ and $\mathbb{H}_{(0,a)} := \{s \in \mathbb{C} : 0 < \sigma < a\}$.

For $n, m \in \mathbb{N}$, we write (n, m) and $[n, m]$ to denote the greatest common divisor and the least common multiple of n and m. C_r^n stands for $\frac{n!}{r!(n-r)!}$.

N will be a parameter which tends to infinity. $\mathfrak{L}$ always denotes a k-tuple $\{a_1n+b_1, \ldots, a_kn+b_k\}$. All implied constants in $\mathcal{O}$ and $\ll$ will be absolute or depend on $\mathfrak{L}$, unless otherwise specified.

ϖ is the characteristic function of prime numbers. μ and ϕ denote the Möbius function and the Euler totient function respectively. ζ denotes the Riemann zeta function. The arithmetic function ω is defined by $\omega(m) = \sum_{p|m} 1$, namely the number of distinct prime factors of m.

X_q is the set of all Dirichlet characters modulo q. X_q^* is the set of all primitive characters in X_q.

For any $i \in \mathbb{N}$ and any symbol v, we shall write $\mathbf{v_i} = (v_1, \ldots, v_i)$. We shall also use the notation $\mathbf{v_0}$. In this case its meaning should be clear from the context. Any empty product is equal to 1, so in particular $p_1 \cdots p_r$=1 when $r = 0$.

The space of functions with continuous partial derivatives of every order on their domains is denoted by $\mathcal{C}^\infty$, and functions therein are called $\mathcal{C}^\infty$-functions. A function with zero variable is treated as a constant.

Finally, we use $\mathbb{A}$ to denote some large constant, which may differ in each occurrence. Similarly, we use c to denote some sufficiently small constant, whose value may vary each time.

Chapter 2

Preliminaries

2.1 HB-admissibility

Given an admissible k-tuple $\mathfrak{L} = \{L_1, L_2, \ldots, L_k\}$, where $L_i(n) = a_i n + b_i$, we begin by making the restriction that n only comes from a certain arithmetic progression. This idea occurs in [10]. In this way, $\mathfrak{L}$ is transformed into another admissible k-tuple which possesses a number of nice properties, thus simplifying our subsequent argument.

Let

$$H = \left(\prod_{1 \le i \le k} a_i \right) \left(\prod_{1 \le i < j \le k} |a_i b_j - a_j b_i| \right).$$

Note that $H \neq 0$. For each $p|H$, since $\mathfrak{L}$ is admissible, there exists m_p such that $p \nmid \prod_{i=1}^{k} L_i(m_p)$; and by the Chinese Remainder Theorem, there exists $B \in \mathbb{N}$ such that $B \equiv m_p (\text{mod } p)$ for all such p. Our restriction is that $n = Hm + B$, and this results in the k-tuple $\mathfrak{L}^*$ with $a_i^* = a_i H$ and $b_i^* = a_i B + b_i$.

Observe that $b_i^* = a_i B + b_i \equiv a_i m_p + b_i (\text{mod } p)$ for each $p|H$, so we have $(b_i^*, H) = 1$. It follows that $(b_i^*, \prod_{j=1}^{k} a_j^*) = 1$, and that the congruence $L_i^*(n) \equiv 0$ (mod p) has no solution for any $p|H$. On the other hand, for each $p_0 \nmid H$, we have $p_0 \nmid a_i^*$, so the congruent $L_i^*(n) \equiv 0 (\text{mod } p_0)$ has a unique solution modulo p_0. It is not possible to solve

$$\begin{cases} L_i^*(n) \equiv 0 \pmod{p_0} \\ L_j^*(n) \equiv 0 \pmod{p_0} \end{cases}$$

simultaneously for distinct i and j, for otherwise

$$\begin{aligned} \begin{cases} a_i^* n_0 + b_i^* \equiv 0 \pmod{p_0} \\ a_j^* n_0 + b_j^* \equiv 0 \pmod{p_0} \end{cases} &\Rightarrow \ a_i^* b_j^* - a_j^* b_i^* \equiv 0 \ (\text{mod } p_0) \\ &\Rightarrow \ a_i b_j^* - a_j b_i^* \equiv 0 \ (\text{mod } p_0) \qquad (\text{by } p_o \nmid H) \\ &\Rightarrow \ a_i b_j - a_j b_i \equiv 0 \ (\text{mod } p_0) \\ &\Rightarrow \ p_0 | H, \ \text{a contradiction.} \end{aligned}$$

Therefore,

$$\nu_p(\mathfrak{L}^*) = \begin{cases} k & \text{if } p \nmid H \\ 0 & \text{otherwise.} \end{cases}$$

We now show that $\mathfrak{L}^*$ is an admissible k-tuple. This follows immediately from the following lemma.

Lemma 2.1.1

If $p \nmid H$, then $p > k$.

Proof

Suppose $p \nmid H$. Then by the definition of H, we have

$$p \nmid \left(\prod_{1 \le i \le k} a_i \right) \left(\prod_{1 \le i < j \le k} |a_i b_j - a_j b_i| \right).$$

Since $p \nmid \prod_{i=1}^k a_i$, for each i the congruence $L_i(n) \equiv 0 (\bmod\ p)$ has a unique solution modulo p. If the congruences $L_i(n) \equiv 0(\bmod\ p)$ and $L_j(n) \equiv 0(\bmod\ p)$ can be solved simultaneously, then we have $a_i b_j - a_j b_i \equiv 0(\bmod\ p)$, which is impossible unless $i = j$. Hence, we have $\nu_p(\mathfrak{L}) = k$. Since $\mathfrak{L}$ is admissible, it follows that $p > k$.

Q.E.D.

Definition 2.1.2

*A k-tuple $\mathfrak{L} = \{a_1 n + b_1, \ldots, a_k n + b_k\}$ is called **HB-admissible** if it is admissible, and by letting*

$$A := \prod_{i=1}^{k} a_i,$$

the following hold:

(i) *$a_1, \ldots, a_k$ are all composed of the same primes, and $(A, \prod_{i=1}^k b_i) = 1$.*

(ii) *If $p \nmid A$, then $p > k$ and $\nu_p(\mathfrak{L}) = k$; else if $p | A$, then $\nu_p(\mathfrak{L}) = 0$.*

(iii) *For $i \ne j$, all prime factors of $a_i b_j - a_j b_i$ divide A.*

Our previous discussion shows that $\mathfrak{L}^*$ satisfies (i) and (ii). Noting that $a_i^* b_j^* - a_j^* b_i^* = H(a_i b_j - a_j b_i)$, we see that $\mathfrak{L}^*$ is HB-admissible. Thus, from now on we are allowed to assume that the admissible k-tuple under consideration is HB-admissible. It also follows from Definition 1.1.3 that for a HB-admissible k-tuple $\mathfrak{L}$,

$$\mathfrak{S}(\mathfrak{L}) = \prod_{p|A} \left(1 - \frac{1}{p}\right)^{-k} \prod_{p \nmid A} \left\{ \left(1 - \frac{1}{p}\right)^{-k} \left(1 - \frac{k}{p}\right) \right\}.$$

We close this section by the following proposition:

Proposition 2.1.3

Let $\{a_1n+b_1, \ldots, a_kn+b_k\}$ be a HB-admissible k-tuple. Then for any $n \in \mathbb{N}$, we have

(a) $(a_in+b_i, a_jn+b_j) = 1$ *for* $i \neq j$*; and*

(b) $(a_in+b_i, A) = 1$ *for all* $1 \leq i \leq k$.

Proof

(a) If $p|a_in+b_i$ and $p|a_jn+b_j$, then $p|[(a_in+b_i)a_j - (a_jn+b_j)a_i]$, which implies $p|[b_ia_j - b_ja_i]$, so $p|a_i$ by condition(iii) and (i) in Definition 2.1.2. It then follows from $p|a_in+b_i$ that $p|b_i$. This contradicts condition(i), namely $(a_i, b_i) = 1$.

(b) Suppose $p|a_in+b_i$ and $p|A$. Since $p|A \Rightarrow p|a_i$, we have $p|b_i$ as well, contradicting $(A, b_i) = 1$.

Q.E.D.

2.2 Asymptotic formulas for Q: Initial Step

Let $\lambda^{\mathbf{D}}_{\delta_1,\ldots,\delta_k}$ be a real-valued arithmetic function in $\delta_1, \ldots, \delta_k$, where $\mathbf{D}$ is a parameter. We assume that $\lambda^{\mathbf{D}}_{\delta_1,\ldots,\delta_k} = 0$ if any $\delta_i > S$, where the support $S = S(N)$ is a function in N.

I. Preparation for Q_0

Suppose $\mathfrak{L}$ is HB-admissible. We would like to find an asymptotic formula for the following sum:

$$Q_0 := \sum_{N<n\leq 2N} \left(\sum_{\substack{\delta_1|a_1n+b_1 \\ \vdots \\ \delta_k|a_kn+b_k}} \lambda^{\mathbf{D}}_{\delta_1,\ldots,\delta_k} \right) \left(\sum_{\substack{\widetilde{\delta}_1|a_1n+b_1 \\ \vdots \\ \widetilde{\delta}_k|a_kn+b_k}} \lambda^{\widetilde{\mathbf{D}}}_{\widetilde{\delta}_1,\ldots,\widetilde{\delta}_k} \right).$$

By changing the order of summation, we get

$$Q_0 = \sum_{\substack{\delta_1,\ldots,\delta_k \\ \widetilde{\delta}_1,\ldots,\widetilde{\delta}_k}} \lambda\widetilde{\lambda} \sum_{\substack{N<n\le 2N \\ \Delta_1|a_1n+b_1 \\ \vdots \\ \Delta_k|a_kn+b_k}} 1,$$

where we write for brevity

$$\lambda := \lambda^{\mathbf{D}}_{\delta_1,\ldots,\delta_k}, \quad \widetilde{\lambda} := \lambda^{\widetilde{\mathbf{D}}}_{\widetilde{\delta}_1,\ldots,\widetilde{\delta}_k}, \quad \text{and} \quad \Delta_i := [\delta_i, \widetilde{\delta}_i].$$

By Proposition 2.1.3, the outer sum can be restricted to $(\Delta_i, \Delta_j) = 1$ and $(\Delta_1 \cdots \Delta_k, A) = 1$. By $(\Delta_1 \cdots \Delta_k, A) = 1$ and the fact that all a_i's and A are composed of the same primes, we see that there exists $m_i \in \mathbb{N}$ such that $\Delta_i | a_in+b_i \Leftrightarrow n \equiv m_i (\bmod\ \Delta_i)$. Since $(\Delta_i, \Delta_j) = 1$, we have, by Chinese Remainder Theorem, there exists $m_0 \in \mathbb{N}$ such that

$$Q_0 = \sum_{\substack{\delta_1,\ldots,\delta_k \\ \widetilde{\delta}_1,\ldots,\widetilde{\delta}_k \\ (\Delta_i,\Delta_j)=1 \\ (\Delta_1\cdots\Delta_k,A)=1}} \lambda\widetilde{\lambda} \sum_{\substack{N<n\le 2N \\ n\equiv m_0 (\bmod\ \Delta_1\cdots\Delta_k)}} 1 = \sideset{}{^\sharp}\sum_{\delta_i,\widetilde{\delta}_i} \lambda\widetilde{\lambda} \frac{N}{\Delta_1\cdots\Delta_k} + \mathcal{O}\left(\sideset{}{^\sharp}\sum_{\delta_i,\widetilde{\delta}_i} |\lambda\widetilde{\lambda}| \right),$$

where we use $\sideset{}{^\sharp}\sum_{\delta_i,\widetilde{\delta}_i}$ to denote the sum $\sum_{\substack{\delta_1,\ldots,\delta_k \\ \widetilde{\delta}_1,\ldots,\widetilde{\delta}_k \\ (\Delta_i,\Delta_j)=1 \\ (\Delta_1\cdots\Delta_k,A)=1}}$.

II. Preparation for Q_j

Without loss of generality, we would like to find an asymptotic formula for

$$Q_j := \sum_{N<n\le 2N} \gamma_j(a_1n+b_1) \left(\sum_{\substack{\delta_i|a_in+b_i \\ (1\le i\le k)}} \lambda^{\mathbf{D}}_{\delta_1,\ldots,\delta_k} \right) \left(\sum_{\substack{\widetilde{\delta}_i|a_in+b_i \\ (1\le i\le k)}} \lambda^{\widetilde{\mathbf{D}}}_{\widetilde{\delta}_1,\ldots,\widetilde{\delta}_k} \right),$$

where $\gamma_j : \mathbb{N} \to \{0, 1\}$ satisfies $\gamma_j(m) = 0$ if m is not an E_j-number. The function γ_j will be specified in the subsequent chapters.

By changing the order of summation,

$$\begin{aligned} Q_j &= \sideset{}{^\sharp}\sum_{\delta_i,\widetilde{\delta}_i} \lambda\widetilde{\lambda} \sum_{\substack{N<n\le 2N \\ \Delta_i|a_in+b_i (1\le i\le k)}} \gamma_j(a_1n+b_1) \\ &= \sum_{r=0}^{j} \sideset{}{^\sharp}\sum_{\substack{\delta_i,\widetilde{\delta}_i \\ \omega(\Delta_1)=r}} \lambda\widetilde{\lambda} \sum_{\substack{N<n\le 2N \\ \Delta_i|a_in+b_i (1\le i\le k)}} \gamma_j(a_1n+b_1). \end{aligned}$$

Let $m = a_1n+b_1$. By $(\Delta_2\cdots\Delta_k, A) = 1$ and the fact that all a_i's and A are composed of the same primes, we have, for any $i \geq 2$, there exists m_i, m_i' such that

$$\Delta_i|a_in+b_i \Leftrightarrow n \equiv m_i \pmod{\Delta_i} \Leftrightarrow m \equiv m_i' \pmod{\Delta_i}.$$

It follows that

$$Q_j = \sum_{r=0}^{j} \sideset{}{^\sharp}\sum_{\substack{\delta_i,\widetilde{\delta}_i \\ \omega(\Delta_1)=r}} \lambda\widetilde{\lambda} \sum_{\substack{a_1N+b_1<m\leq 2a_1N+b_1 \\ m\equiv b_1 (\mathrm{mod}\ a_1) \\ m\equiv m_i' (\mathrm{mod}\ \Delta_i)(2\leq i\leq k) \\ \Delta_1|m}} \gamma_j(m).$$

We now make two assumptions about the support of λ and γ_j. Firstly, we suppose that $S \leq N^{1/2}$. With this, when $r = j$ so that $\omega(\Delta_1) = j$, since $\Delta_1 \leq \delta_1\widetilde{\delta}_1 \leq S^2 \leq N$, we see that the sum over m vanishes. Secondly, we suppose that $\gamma_j(m) = 0$ if the least prime factor of m is not greater than $Y = Y(N)$.

Under these assumptions, we have

$$Q_j = \sum_{r=0}^{j-1} \sum_{\substack{Y<p_1<\cdots<p_r \\ (p_1\cdots p_r,A)=1}} \sideset{}{^\sharp}\sum_{\substack{\delta_i,\widetilde{\delta}_i \\ \Delta_1=p_1\cdots p_r}} \lambda\widetilde{\lambda} \sum_{\substack{a_1N+b_1<m\leq 2a_1N+b_1 \\ m\equiv b_1 (\mathrm{mod}\ a_1) \\ m\equiv m_i' (\mathrm{mod}\ \Delta_i)(2\leq i\leq k) \\ p_1\cdots p_r|m}} \gamma_j(m). \tag{2.1}$$

By the Chinese Remainder Theorem, there exists m_0 such that

$$\begin{cases} m \equiv b_1 \pmod{a_1} \\ m \equiv m_i' \pmod{\Delta_i}(2 \leq i \leq k) \end{cases} \Leftrightarrow m \equiv m_0 \pmod{a_1\Delta_2\cdots\Delta_k}.$$

On the other hand, since $(p_1\cdots p_r, a_1\Delta_2\cdots\Delta_k) = 1$, there exist q_i's such that $p_iq_i \equiv 1 (\mathrm{mod}\ a_1\Delta_2\cdots\Delta_k)$. Hence, the innermost sum in (2.1) is

$$\begin{aligned} &= \sum_{\substack{a_1N+b_1<m\leq 2a_1N+b_1 \\ m\equiv m_0 (\mathrm{mod}\ a_1\Delta_2\cdots\Delta_k) \\ p_1\cdots p_r|m}} \gamma_j(m) = \sum_{\substack{\frac{a_1N+b_1}{p_1\cdots p_r}<\ell\leq \frac{2a_1N+b_1}{p_1\cdots p_r} \\ \ell\equiv m_0q_1\cdots q_r (\mathrm{mod}\ a_1\Delta_2\cdots\Delta_k)}} \gamma_j(\ell p_1\cdots p_r) \\ &\qquad = \sum_{\substack{\frac{a_1N}{p_1\cdots p_r}<\ell\leq \frac{2a_1N}{p_1\cdots p_r} \\ \ell\equiv m_0q_1\cdots q_r (\mathrm{mod}\ a_1\Delta_2\cdots\Delta_k)}} \gamma_j(\ell p_1\cdots p_r) + \mathcal{O}(1)\,. \end{aligned}$$

For $p_1, \ldots, p_r$ with $Y < p_1 < \cdots < p_r$, let $\gamma_j(\,\cdot\,|p_1,\cdots,p_r) : \mathbb{N} \to [0,1]$ be a function such that, for any $\ell \in [\frac{N}{p_1\cdots p_r}, \frac{2AN}{p_1\cdots p_r}]$, we have $\gamma_j(\ell|p_1,\cdots,p_r) = \gamma_j(\ell p_1\cdots p_r)$ if $(\ell, p_1\cdots p_r) = 1$. In particular, when $r = 0$, we have $\gamma_j(\ell|p_1,\cdots,p_r) = \gamma_j(\ell)$ for $\ell \in [N, 2AN]$.

Write $\mathbf{p_r} = (p_1, \cdots, p_r)$. The main term in the last expression becomes

$$= \sum_{\substack{\frac{a_1N}{p_1\cdots p_r} < \ell \le \frac{2a_1N}{p_1\cdots p_r} \\ \ell \equiv m_0 q_1\cdots q_r (\mathrm{mod}\ a_1\Delta_2\cdots\Delta_k)}} \gamma_j(\ell|\mathbf{p_r}) + \mathcal{O}\left(\sum_{\substack{\frac{a_1N}{p_1\cdots p_r} < \ell \le \frac{2a_1N}{p_1\cdots p_r} \\ \ell \equiv m_0 q_1\cdots q_r (\mathrm{mod}\ a_1\Delta_2\cdots\Delta_k) \\ (\ell, p_1\cdots p_r) > 1}} 1 \right).$$

The $\mathcal{O}$-term above is trivial when $r = 0$. When $r > 0$, it is

$$\begin{aligned}
&\ll \sum_{\substack{\frac{a_1N}{p_1\cdots p_r\cdot p_1} < \ell \le \frac{2a_1N}{p_1\cdots p_r \cdot p_1} \\ \ell \equiv m_0 q_1\cdots q_r\cdot q_1 (\mathrm{mod}\ a_1\Delta_2\cdots\Delta_k)}} 1 + \cdots + \sum_{\substack{\frac{a_1N}{p_1\cdots p_r\cdot p_r} < \ell \le \frac{2a_1N}{p_1\cdots p_r \cdot p_r} \\ \ell \equiv m_0 q_1\cdots q_r\cdot q_r (\mathrm{mod}\ a_1\Delta_2\cdots\Delta_k)}} 1 \\
&\ll \frac{N}{p_1^2\cdots p_r}\frac{1}{\Delta_2\cdots\Delta_k} + \cdots + \frac{N}{p_1\cdots p_r^2}\frac{1}{\Delta_2\cdots\Delta_k} + 1 \\
&\ll \frac{N}{Yp_1\cdots p_r}\frac{1}{\Delta_2\cdots\Delta_k} + 1,
\end{aligned}$$

by noting that $Y < p_1 < \cdots < p_r$.

Let

$$\begin{aligned}
\mathcal{E}_{\gamma_j(\cdot|\mathbf{p_r})}(x;q,a) &= \sum_{\substack{x<\ell\le 2x \\ \ell\equiv a (\mathrm{mod}\ q)}} \gamma_j(\ell|\mathbf{p_r}) - \frac{1}{\phi(q)} \sum_{\substack{x<\ell\le 2x \\ (\ell,q)=1}} \gamma_j(\ell|\mathbf{p_r}), \\
\mathcal{E}^*_{\gamma_j(\cdot|\mathbf{p_r})}(x;q) &= \max_{(a,q)=1} |\mathcal{E}_{\gamma_j(\cdot|\mathbf{p_r})}(x;q,a)|.
\end{aligned}$$

Then the innermost sum in (2.1) becomes

$$\begin{aligned}
\frac{1}{\phi(a_1\Delta_2\cdots\Delta_k)} \sum_{\substack{\frac{a_1N}{p_1\cdots p_r} < \ell \le \frac{2a_1N}{p_1\cdots p_r} \\ (\ell, a_1\Delta_2\cdots\Delta_k)=1}} \gamma_j(\ell|\mathbf{p_r}) \quad &+ \quad \mathcal{O}\left(\mathcal{E}^*_{\gamma_j(\cdot|\mathbf{p_r})}\left(\frac{a_1N}{p_1\cdots p_r}; a_1\Delta_2\cdots\Delta_k\right)\right) \\
&+ \quad \mathcal{O}\left(\frac{N}{Yp_1\cdots p_r}\frac{1}{\Delta_2\cdots\Delta_k}\right) + \mathcal{O}(1).
\end{aligned}$$

We conclude that

$$
\begin{aligned}
Q_j &= \sum_{r=0}^{j-1} \sum_{\substack{Y<p_1<\cdots<p_r\\(p_1\cdots p_r,A)=1}} \sideset{}{^\sharp}\sum_{\substack{\delta_i,\widetilde{\delta}_i\\\Delta_1=p_1\cdots p_r}} \frac{\lambda\widetilde{\lambda}}{\phi(a_1\Delta_2\cdots\Delta_k)} \sum_{\substack{\frac{a_1N}{p_1\cdots p_r}<\ell\leq\frac{2a_1N}{p_1\cdots p_r}\\(\ell,a_1\Delta_2\cdots\Delta_k)=1}} \gamma_j(\ell|\mathbf{p_r})\\
&+\mathcal{O}\left(\sum_{r=0}^{j-1} \sum_{Y<p_1<\cdots<p_r} \sideset{}{^\sharp}\sum_{\substack{\delta_i,\widetilde{\delta}_i\\\Delta_1=p_1\cdots p_r}} |\lambda\widetilde{\lambda}|\ \mathcal{E}^*_{\gamma_j(\cdot|\mathbf{p_r})}\left(\frac{a_1N}{p_1\cdots p_r};a_1\Delta_2\cdots\Delta_k\right)\right)\\
&+\mathcal{O}\left(\sum_{r=1}^{j-1} \sum_{Y<p_1<\cdots<p_r} \sideset{}{^\sharp}\sum_{\substack{\delta_i,\widetilde{\delta}_i\\\Delta_1=p_1\cdots p_r}} \frac{|\lambda\widetilde{\lambda}|N}{Yp_1\cdots p_r}\frac{1}{\Delta_2\cdots\Delta_k}\right)+\mathcal{O}\left(\sideset{}{^\sharp}\sum_{\delta_i,\widetilde{\delta}_i}|\lambda\widetilde{\lambda}|\right).
\end{aligned}
$$

Finally, we make two remarks:

(i) Write λ_{δ_1} for $\lambda^{\mathbf{D}}_{\delta_1,\ldots,\delta_k}$, $\widetilde{\lambda}_{\widetilde{\delta}_1}$ for $\lambda^{\widetilde{\mathbf{D}}}_{\widetilde{\delta}_1,\ldots,\widetilde{\delta}_k}$. The sums over $p_1,\cdots,p_r$ can be restricted to those $p_1\cdots p_r=[\delta_1,\widetilde{\delta}_1]$ such that $\lambda_{\delta_1}\widetilde{\lambda}_{\widetilde{\delta}_1}\neq 0$

(ii) By changing the order of summation, the main term in the formula above is

$$
=\sum_{r=0}^{j-1} \sum_{Y<p_1<\cdots<p_r} \sum_{\substack{\frac{a_1N}{p_1\cdots p_r}<\ell\leq\frac{2a_1N}{p_1\cdots p_r}\\(p_1\cdots p_r\ell,A)=1}} \gamma_j(\ell|\mathbf{p_r}) \sum_{\substack{\delta_2,\ldots,\delta_k\\\widetilde{\delta}_2,\ldots,\widetilde{\delta}_k\\(\Delta_i,\Delta_j)=1\\(\Delta_2\cdots\Delta_k,Ap_1\cdots p_r\ell)=1}} \frac{\left[\sum_{[\delta_1,\widetilde{\delta}_1]=p_1\cdots p_r}\lambda_{\delta_1}\widetilde{\lambda}_{\widetilde{\delta}_1}\right]}{\phi(a_1\Delta_2\cdots\Delta_k)}.
$$

III. Summary and Transition

We first summarize the results obtained in this section.

<u>**Definition 2.2.1**</u>

For any arithmetic function h, we define

$$
\begin{aligned}
\mathcal{E}_h(x;q,a) &:= \sum_{\substack{x<\ell\leq 2x\\\ell\equiv a(\mathrm{mod}\ q)}} h(\ell)-\frac{1}{\phi(q)}\sum_{\substack{x<\ell\leq 2x\\(\ell,q)=1}} h(\ell),\\
\mathcal{E}^*_h(x;q) &:= \max_{(a,q)=1}|\mathcal{E}_h(x;q,a)|.
\end{aligned}
$$

Theorem 2.2.2

Let $\mathfrak{L}$ be a HB-admissible k-tuple, and let $\lambda^{\mathbf{D}}_{\delta_1,\dots,\delta_k}$ be a real-valued arithmetic function in $\delta_1,\dots,\delta_k$, where $\mathbf{D}$ is a parameter. Suppose $\lambda^{\mathbf{D}}_{\delta_1,\dots,\delta_k}=0$ if any $\delta_i>S$, where $S=S(N)$ is a function in N. Write $\lambda=\lambda_{\delta_1}=\lambda^{\mathbf{D}}_{\delta_1,\dots,\delta_k}$, $\widetilde{\lambda}=\widetilde{\lambda}_{\widetilde{\delta}_1}=\lambda^{\widetilde{\mathbf{D}}}_{\widetilde{\delta}_1,\dots,\widetilde{\delta}_k}$, and $\Delta_i=[\delta_i,\widetilde{\delta}_i]$.

(a) *We have*

$$\sum_{N<n\le 2N}\left(\sum_{\delta_i|a_in+b_i}\lambda^{\mathbf{D}}_{\delta_1,\dots,\delta_k}\right)\left(\sum_{\widetilde{\delta}_i|a_in+b_i}\lambda^{\widetilde{\mathbf{D}}}_{\widetilde{\delta}_1,\dots,\widetilde{\delta}_k}\right)$$

$$=\sum_{\substack{\delta_1,\dots,\delta_k\\ \widetilde{\delta}_1,\dots,\widetilde{\delta}_k\\ (\Delta_i,\Delta_j)=1\\ (\Delta_1\cdots\Delta_k,A)=1}}\lambda\widetilde{\lambda}\frac{N}{\Delta_1\cdots\Delta_k}+\mathcal{O}\left(\sum_{\substack{\delta_1,\dots,\delta_k\\ \widetilde{\delta}_1,\dots,\widetilde{\delta}_k}}|\lambda\widetilde{\lambda}|\right).$$

(b) *Suppose $S\le N^{1/2}$. Let $j\ge 1$ and suppose $\gamma_j:\mathbb{N}\to\{0,1\}$ satisfies $\gamma_j(m)=0$ if m is not an E_j-number, or if the least prime factor of m is not greater than $Y=Y(N)$.*

Also, for any $0\le r\le j-1$, and any $p_1,\dots,p_r$ with $Y<p_1<\cdots<p_r$, suppose $\gamma_j(\,\cdot\,|\mathbf{p_r}):\mathbb{N}\to[0,1]$ is a function such that, for any $\ell\in[\frac{N}{p_1\cdots p_r},\frac{2AN}{p_1\cdots p_r}]$, we have $\gamma_j(\ell|\mathbf{p_r})=\gamma_j(\ell p_1\cdots p_r)$ if $(\ell,p_1\cdots p_r)=1$.

Then

$$\sum_{N<n\le 2N}\gamma_j(a_1n+b_1)\left(\sum_{\delta_i|a_in+b_i}\lambda^{\mathbf{D}}_{\delta_1,\dots,\delta_k}\right)\left(\sum_{\widetilde{\delta}_i|a_in+b_i}\lambda^{\widetilde{\mathbf{D}}}_{\widetilde{\delta}_1,\dots,\widetilde{\delta}_k}\right)$$

$$=\sum_{r=0}^{j-1}\sum_{Y<p_1<\cdots<p_r}\sum_{\substack{\frac{a_1N}{p_1\cdots p_r}<\ell\le\frac{2a_1N}{p_1\cdots p_r}\\ (p_1\cdots p_r\ell,A)=1}}\gamma_j(\ell|\mathbf{p_r})\sum_{\substack{\delta_2,\dots,\delta_k\\ \widetilde{\delta}_2,\dots,\widetilde{\delta}_k\\ (\Delta_i,\Delta_j)=1\\ (\Delta_2\cdots\Delta_k,Ap_1\cdots p_r\ell)=1}}\frac{\left[\sum_{[\delta_1,\widetilde{\delta}_1]=p_1\cdots p_r}\lambda_{\delta_1}\widetilde{\lambda}_{\widetilde{\delta}_1}\right]}{\phi(a_1\Delta_2\cdots\Delta_k)}$$

$$+\mathcal{O}\left(\sum_{r=0}^{j-1}\sum_{Y<p_1<\cdots<p_r}\sum_{\substack{\delta_1,\dots,\delta_k\\ \widetilde{\delta}_1,\dots,\widetilde{\delta}_k\\ (\Delta_i,\Delta_j)=1\\ (\Delta_1\cdots\Delta_k,A)=1\\ \Delta_1=p_1\cdots p_r}}|\lambda\widetilde{\lambda}|\mathcal{E}^*_{\gamma_j(\cdot|\mathbf{p_r})}\left(\frac{a_1N}{p_1\cdots p_r};a_1\Delta_2\cdots\Delta_k\right)\right)$$

$$+\mathcal{O}\left(\sum_{r=1}^{j-1}\sum_{Y<p_1<\cdots<p_r}\sum_{\substack{\delta_1,\ldots,\delta_k\\ \widetilde{\delta}_1,\ldots,\widetilde{\delta}_k\\ (\Delta_i,\Delta_j)=1\\ \Delta_1=p_1\cdots p_r}}\frac{|\lambda\widetilde{\lambda}|N}{Yp_1\cdots p_r}\frac{1}{\Delta_2\cdots\Delta_k}\right)+\mathcal{O}\left(\sum_{\substack{\delta_1,\ldots,\delta_k\\ \widetilde{\delta}_1,\ldots,\widetilde{\delta}_k}}|\lambda\widetilde{\lambda}|\right),$$

where the sums over $p_1,\cdots,p_r$ *are restricted to those* $p_1\cdots p_r=[\delta_1,\widetilde{\delta}_1]$ *with* $\lambda_{\delta_1}\widetilde{\lambda}_{\widetilde{\delta}_1}\neq 0$.

We have set up a general framework for subsequent work. In the remaining parts of this chapter, a number of preliminary lemmas will be established. We shall collect some basic results in the next section. In Section 2.4, we shall deal with various iterated objects. Finally, we shall prove a variation of Bombieri-Vinogradov Theorem in Section 2.5. The reader may skip the material until they find themselves necessary to refer back.

2.3 Basic results

Lemma 2.3.1 (This is an exercise in [14, chapter 7])

Let $q\in\mathbb{N}$ *be square-free. Then there are exactly* $3^{\omega(q)}$ *pairs of* q_1, q_2 *such that* $[q_1,q_2]=q$.

Proof

All q_1, q_2 such that $[q_1,q_2]=q$ are square-free and composed of some of the primes in q. They can be constructed by requiring, for each $p|q$, exactly one of the following holds:

(i) $p|q_1$ and $p\nmid q_2$ (ii) $p\nmid q_1$ and $p|q_2$ (iii) $p|q_1$ and $p|q_2$.

It follows that the total number is $\prod_{p|q}3=3^{\omega(q)}$.

Q.E.D.

Lemma 2.3.2

For $h\in\mathbb{N}$, *we have* $\displaystyle\sum_{q<N}\frac{\mu^2(q)h^{\omega(q)}}{q}\ll\log^h N$, *where the implied constant only depends on* h.

Proof

By Merten's formula, we have $\prod_{p<N}\left(1+\frac{1}{p}\right) \ll \log N$. Hence

$$\sum_{q<N} \frac{\mu^2(q)h^{\omega(q)}}{q} \leq \prod_{p<N}\left(1+\frac{h}{p}\right) \leq \prod_{p<N}\left(1+\frac{1}{p}\right)^h \ll \log^h N.$$

Q.E.D.

Lemma 2.3.3

(a) *There exist* $c_0, t_0 > 0$ *such that* $\zeta(s) = \mathcal{O}(\log|t|)$, $\frac{1}{\zeta(s)} = \mathcal{O}(\log|t|)$ *in the region* $1 - \frac{100c_0}{\log|t|} \leq \sigma \leq 2, |t| > t_0$.

(b) *For the* t_0 *that occurs in* (a), *we have* $\zeta(s) = \mathcal{O}\left(\frac{1}{|s-1|}\right)$ *in the region* $0 \leq \sigma \leq 2, |t| \leq t_0$.

Proof

(a) It follows from the results in [18, section 3.5 and 3.11] and the fact that $\zeta(\bar{s}) = \overline{\zeta(s)}$.

(b) It follows from the fact that ζ is a meromorphic function with a single simple pole at $s = 1$.

Q.E.D.

Lemma 2.3.4

Let $a \in \mathbb{R}$, $a \neq 0$ *and* $n \in \mathbb{N}$. *Then*

(i) $\displaystyle\int_{\{a+it:-T\leq t\leq T\}} \frac{|ds|}{|s|} \ll \frac{1}{|a|} + \log T$ *for* $T > T_0$; *and*

(ii) $\displaystyle\int_{(a)} \frac{|ds|}{|s|^n} \ll \frac{1}{|a|^{n-1}}$ *if* $n \geq 2$, *where the implied constant only depends on* n.

Proof

(i)

$$\int\limits_{\{a+it:-T\le t\le T\}} \frac{|ds|}{|s|} = \int\limits_{-T}^{T} \frac{1}{(a^2+t^2)^{\frac{1}{2}}}dt \le \int\limits_{-1}^{1} \frac{1}{|a|}du + 2\int\limits_{1}^{T} \frac{1}{t}dt \ll \frac{1}{|a|} + \log T.$$

(ii)

$$\int\limits_{(a)} \frac{|ds|}{|s|^n} = \int\limits_{-\infty}^{\infty} \frac{1}{(a^2+t^2)^{\frac{n}{2}}}dt = \frac{1}{|a|^n}\int\limits_{-\infty}^{\infty} \frac{|a|}{(1+u^2)^{\frac{n}{2}}}du \ll \frac{1}{|a|^{n-1}}.$$

Q.E.D.

Lemma 2.3.5

Let $\ell, \kappa \in \mathbb{N}$; $p_0, \varphi > 0$; $b = 0$ or 1. Then there exists $a = a(\ell, \kappa, \varphi) > 0$ such that $P : \prod_{i=1}^{2\kappa} \mathbb{H}_{-a} \to \mathbb{C}$,

$$P(s_1, \widetilde{s_1}, \ldots, s_\kappa, \widetilde{s_\kappa}) = \prod_{p>p_0} \left\{ \begin{array}{l} \left[1 - \frac{1}{p-b}\sum_{i=1}^{\kappa}\left(\frac{\ell}{p^{\varphi s_i}} + \frac{\ell}{p^{\varphi \widetilde{s_i}}} - \frac{\ell}{p^{\varphi s_i + \varphi \widetilde{s_i}}}\right)\right] \times \\ \prod_{i=1}^{\kappa}\left(1 - \frac{1}{p^{1+\varphi s_i}}\right)^{-\ell}\left(1 - \frac{1}{p^{1+\varphi \widetilde{s_i}}}\right)^{-\ell}\left(1 - \frac{1}{p^{1+\varphi s_i + \varphi \widetilde{s_i}}}\right)^{\ell} \end{array} \right\}$$

is defined, the convergence is uniform, and P is bounded.

Proof

Let $\mathcal{O}_p = \mathcal{O}(p^{-1.5})$, where the implied constant only depends on ℓ, κ and φ. First, note that for all σ_i, $\widetilde{\sigma_i} > -c$, the factor in the square brackets is

$$1 - \sum_{i=1}^{\kappa}\left(\frac{\ell}{p^{1+\varphi s_i}} + \frac{\ell}{p^{1+\varphi \widetilde{s_i}}} - \frac{\ell}{p^{1+\varphi s_i + \varphi \widetilde{s_i}}}\right) + \mathcal{O}_p.$$

On the other hand, for σ_i, $\widetilde{\sigma_i} > -c$, we have

$$\begin{aligned} &\prod_{i=1}^{\kappa}\left(1 - \frac{1}{p^{1+\varphi s_i}}\right)^{-\ell}\left(1 - \frac{1}{p^{1+\varphi \widetilde{s_i}}}\right)^{-\ell}\left(1 - \frac{1}{p^{1+\varphi s_i + \varphi \widetilde{s_i}}}\right)^{\ell} \\ = &\prod_{i=1}^{\kappa}\left(1 + \frac{\ell}{p^{1+\varphi s_i}} + \mathcal{O}_p\right)\left(1 + \frac{\ell}{p^{1+\varphi \widetilde{s_i}}} + \mathcal{O}_p\right)\left(1 - \frac{\ell}{p^{1+\varphi s_i + \varphi \widetilde{s_i}}} + \mathcal{O}_p\right) \\ = &\prod_{i=1}^{\kappa}\left(1 + \frac{\ell}{p^{1+\varphi s_i}} + \frac{\ell}{p^{1+\varphi \widetilde{s_i}}} - \frac{\ell}{p^{1+\varphi s_i + \varphi \widetilde{s_i}}} + \mathcal{O}_p\right). \end{aligned}$$

We now expand the product above, which is equivalent to choosing κ times one of the five terms in the bracket and then multiplying them altogether. Note that if two or more terms chosen are not equal to 1, then the term produced is $\mathcal{O}_p$. Hence, the expansion results in

$$1+\sum_{i=1}^{\kappa}\left(\frac{\ell}{p^{1+\varphi s_i}}+\frac{\ell}{p^{1+\varphi \widetilde{s_i}}}-\frac{\ell}{p^{1+\varphi s_i+\varphi \widetilde{s_i}}}\right)+\mathcal{O}_p.$$

Therefore, uniformly for all σ_i, $\widetilde{\sigma_i} > -c$, the factor in the curly braces is $1+\mathcal{O}_p$. The results follow.

Q.E.D.

2.4 Iterative matters

Lemma 2.4.1 (Morera's Theorem)

Let G be a region in $\mathbb{C}$ and let $f : G \to \mathbb{C}$ be a continuous function such that $\int_T f = 0$ for every triangular path T in G; then f is analytic in G.

Proof

See [3, IV.5.10].

Q.E.D.

Proposition 2.4.2

Let $f_n : \prod_{i=1}^{\kappa} \mathbb{H}_{-a} \to \mathbb{C}$ be a sequence of functions in $\mathcal{C}^\infty$ such that $f_n \to f$ uniformly on its domain. Then given any $\varepsilon > 0$, and any differential operator of the form

$$\partial_r = \frac{\partial^{m_r}}{\partial s_{i_r}^{m_r}} \cdots \frac{\partial^{m_1}}{\partial s_{i_1}^{m_1}},$$

we have $\partial_r(f)$ exists and $\partial_r(f_n) \to \partial_r(f)$ uniformly on $\prod_{i=1}^{\kappa} \mathbb{H}_{-a+r\varepsilon}$. As a result, we have $f \in \mathcal{C}^\infty$.

Proof (c.f. [3, VII.2.1])

This is proved by induction on r. The case $r = 0$ is trivial.

Suppose now $\partial_r(f)$ exists and $\partial_r(f_n) \to \partial_r(f)$ uniformly on $\prod_{i=1}^{\kappa} \mathbb{H}_{-a+r\varepsilon}$. By Lemma 2.4.1 and the uniform convergence of $\{\partial_r(f_n)\}$, we see that $\partial_{r+1}(f)$ exists on $\prod_{i=1}^{\kappa} \mathbb{H}_{-a+r\varepsilon}$.

Moreover, for any $\mathbf{z}_\kappa \in \prod_{i=1}^{\kappa} \mathbb{H}_{-a+(r+1)\varepsilon}$, let C be the circle $|s_{i_{r+1}} - z_{i_{r+1}}| = \varepsilon$ described in the positive sense. Then

$$\begin{aligned}
& |\partial_{r+1}(f)(\mathbf{z}_\kappa) - \partial_{r+1}(f_n)(\mathbf{z}_\kappa)| \\
= & \left| \frac{m_{r+1}!}{2\pi i} \int_C \frac{(\partial_r(f) - \partial_r(f_n))(z_1, \ldots, s_{i_{r+1}}, \ldots, z_\kappa)}{(s_{i_{r+1}} - z_{i_{r+1}})^{m_{r+1}+1}} ds_{i_{r+1}} \right| \\
\leq & \frac{m_{r+1}!}{2\pi} \frac{2\pi\varepsilon}{\varepsilon^{m_{r+1}+1}} \sup |\partial_r(f) - \partial_r(f_n)|.
\end{aligned}$$

This implies $\partial_{r+1}(f_n) \to \partial_{r+1}(f)$ uniformly on $\prod_{i=1}^{\kappa} \mathbb{H}_{-a+(r+1)\varepsilon}$, and the induction step is complete.

Q.E.D.

Lemma 2.4.3

Suppose $G(s_1, \widetilde{s_1}, \ldots, s_\kappa, \widetilde{s_\kappa})$ is a $\mathcal{C}^\infty$-function on $\prod_{i=1}^{2\kappa} \mathbb{H}_{-a}$, and let

$$f(s_1, \widetilde{s_1}, \ldots, s_\kappa, \widetilde{s_\kappa}) := G(s_1, \widetilde{s_1}, \ldots, s_\kappa, \widetilde{s_\kappa}) \times \left[\prod_{i=1}^{\kappa} \frac{1}{(s_i + \widetilde{s_i})^{\ell_i} s_i^{d_i} \widetilde{s_i}^{\widetilde{d_i}}} \right],$$

where $\ell_i, d_i, \widetilde{d_i} \in \mathbb{N}$. Define

$$R_\emptyset = f, \quad R_{(0)} = \left\{ \operatorname*{Res}_{\widetilde{s_1}=0} \left\{ \operatorname*{Res}_{s_1=0} \{R_\emptyset\} \right\} \right\}, \quad R_{(1)} = \left\{ \operatorname*{Res}_{s_1=-\widetilde{s_1}} \{R_\emptyset\} \right\};$$

and in general, given $v \in \{0,1\}^{j-1}$, we define $R_{(v,i)}$ by

$$R_{(v,i)} = \begin{cases} \left\{ \operatorname*{Res}_{\widetilde{s_j}=0} \left\{ \operatorname*{Res}_{s_j=0} \{R_v\} \right\} \right\} & \text{if } i = 0 \\ \left\{ \operatorname*{Res}_{s_j=-\widetilde{s_j}} \{R_v\} \right\} & \text{if } i = 1. \end{cases}$$

Fix $1 \leq j \leq \kappa$ and $v \in \{0,1\}^{j-1}$. Let $v_{\tau_1}, \ldots v_{\tau_l}$ be all the entries in v having value 1, $v_{\theta_1}, \ldots v_{\theta_\varrho}$ be all the entries having value 0, so that $l + \varrho = j - 1$. Then when we fix $(\widetilde{s_j}, s_{j+1}, \widetilde{s_{j+1}}, \ldots, s_\kappa, \widetilde{s_\kappa}) \in \prod_{i=2j}^{2\kappa} \mathbb{H}_{(0,a)}$, fix $\widetilde{s_\tau} \in \mathbb{H}_{(0,a)}$ for $\tau = \tau_1, \ldots, \tau_l$, and regard R_v as a function of s_j on $\mathbb{H}_{-a}$, the only possible poles of R_v are at $s_j = 0$ and $s_j = -\widetilde{s_j}$.

Furthermore, when we fix $(s_{j+1}, \widetilde{s_{j+1}}, \ldots, s_\kappa, \widetilde{s_\kappa}) \in \prod_{i=2j+1}^{2\kappa} \mathbb{H}_{(0,a)}$, fix $\widetilde{s_\tau} \in \mathbb{H}_{(0,a)}$, and regard $\left\{ \operatorname*{Res}_{s_j=0} \{R_v\} \right\}$ as a function of $\widetilde{s_j}$ on $\mathbb{H}_{-a}$, the only possible pole of it is at $\widetilde{s_j} = 0$.

Proof

This is proved by induction on j.

(i) Suppose $j = 1$. Fix $(\widetilde{s_1}, \ldots, s_\kappa, \widetilde{s_\kappa}) \in \prod_{i=2}^{2\kappa} \mathbb{H}_{(0,a)}$. When we regard $R_\emptyset$ as a function of s_1 on $\mathbb{H}_{-a}$, the only possible poles are at $s_1 = 0$ and $s_1 = -\widetilde{s_1}$, since G is analytic in s_1 on $\mathbb{H}_{-a}$. To find $\left\{\underset{s_1=0}{\mathrm{Res}}\{R_\emptyset\}\right\}$, we consider the series expansion of f at $s_1 = 0$. The series expansion of G is

$$= \left. G\right|_{s_1=0} + s_1 \cdot \left.\frac{\partial G}{\partial s_1}\right|_{s_1=0} + \frac{s_1^2}{2!} \cdot \left.\frac{\partial^2 G}{\partial s_1^2}\right|_{s_1=0} + \cdots$$

The series expansion of $\frac{1}{(s_1+\widetilde{s_1})^{\ell_1}}$ is, by $\frac{1}{(1-s)^n} = \sum_{r=0}^{\infty} \mathrm{C}_r^{n+r-1} s^r$ when $|s| < 1$,

$$= \sum_{r=0}^{\infty} \mathrm{C}_r^{\ell_1+r-1} \frac{(-s_1)^r}{\widetilde{s_1}^{\ell_1+r}}.$$

These expressions, together with the fact that $G \in \mathcal{C}^\infty$, show that when we fix $(s_2, \widetilde{s_2}, \ldots, s_\kappa, \widetilde{s_\kappa}) \in \prod_{i=3}^{2\kappa} \mathbb{H}_{(0,a)}$ and regard $\left\{\underset{s_1=0}{\mathrm{Res}}\{R_\emptyset\}\right\}$ as a function of $\widetilde{s_1}$ on $\mathbb{H}_{-a}$, the only possible pole is at $\widetilde{s_1} = 0$. To find the residue of this pole, namely $R_{(0)}$, we can consider the iterated series expansion of f, first at $s_1 = 0$, and then at $\widetilde{s_1} = 0$. The "coefficient" corresponds to $\frac{1}{\widetilde{s_1} s_1}$ is the required one. Since such an iterated series expansion of G is

$$= \sum_{\widetilde{m_1}}^{\infty} \sum_{m_1}^{\infty} \frac{\widetilde{s_1}^{m_1} s_1^{m_1}}{\widetilde{m_1}!\, m_1!} \left.\frac{\partial^{\widetilde{m_1}+m_1} G}{\partial \widetilde{s_1}^{\widetilde{m_1}} \partial s_1^{m_1}}\right|_{s_1=0, \widetilde{s_1}=0},$$

we see that

$$R_{(0)} = G_{(0)} \left[\prod_{i=2}^{\kappa} \frac{1}{(s_i + \widetilde{s_i})^{\ell_i} s_i^{d_i} \widetilde{s_i}^{\widetilde{d_i}}} \right],$$

where $G_{(0)} = G_{(0)}(s_2, \widetilde{s_2}, \ldots, s_\kappa, \widetilde{s_\kappa})$ is a $\mathcal{C}^\infty$-function on $\prod_{i=1}^{2\kappa-2} \mathbb{H}_{-a}$.

We get $R_{(1)}$ in a similar way. Given $(\widetilde{s_1}, \ldots, s_\kappa, \widetilde{s_\kappa}) \in \prod_{i=2}^{2\kappa} \mathbb{H}_{(0,a)}$, by considering the series expansion of f at $s_1 = -\widetilde{s_1}$, we see that

$$R_{(1)} = G_{(1)} \left[\prod_{i=2}^{\kappa} \frac{1}{(s_i + \widetilde{s_i})^{\ell_i} s_i^{d_i} \widetilde{s_i}^{\widetilde{d_i}}} \right],$$

where when $\widetilde{s_1} \in \mathbb{H}_{(0,a)}$ is fixed, $G_{(1)} = G_{(1)}(s_2, \widetilde{s_2}, \ldots, s_\kappa, \widetilde{s_\kappa}; \widetilde{s_1})$ is a $\mathcal{C}^\infty$-function on $\prod_{i=1}^{2\kappa-2} \mathbb{H}_{-a}$.

(ii) Suppose for $1 \leq j \leq \kappa$, $v \in \{0,1\}^{j-1}$, and $(\widetilde{s_{\tau_1}}, \ldots, \widetilde{s_{\tau_l}}) \in \prod_{i=1}^{l} \mathbb{H}_{(0,a)}$ fixed, we have

$$R_{(v)} = G_{(v)} \left[\prod_{i=j}^{\kappa} \frac{1}{(s_i + \widetilde{s_i})^{\ell_i} s_i^{d_i} \widetilde{s_i}^{\widetilde{d_i}}} \right],$$

where $G_{(v)} = G_{(v)}(s_j, \widetilde{s_j}, \ldots, s_\kappa, \widetilde{s_\kappa}; \widetilde{s_{\tau_1}}, \ldots, \widetilde{s_{\tau_l}})$ is a $\mathcal{C}^\infty$-function on $\prod_{i=1}^{2\kappa-2j+2} \mathbb{H}_{-a}$. Then by exactly the same argument as in (i), we complete the induction step.

Q.E.D.

Lemma 2.4.4

Let $\varphi, b > 0$, and let $w, \kappa \in \mathbb{N}$, $w \geq \kappa$. Suppose that the set $\{1, \ldots, w\}$ is partitioned into a disjoint union

$$\{1, \ldots, w\} = \bigcup_{i=1}^{\kappa} S_i.$$

For any w ordered positive integers $\mathbf{m_w}$, let $M_i = M_i(\mathbf{m_w}) = \prod_{j \in S_i} m_j^{\varphi}$ for $1 \leq i \leq \kappa$. Then for any $d_1, \ldots, d_\kappa \geq 2$, $d_i \in \mathbb{N}$, and any $x_1, \ldots, x_\kappa > 0$, we have

$$\sum_{\substack{m_1, \ldots, m_w = 1 \\ M_i \leq x_i \, (1 \leq i \leq \kappa)}}^{\infty} a_{\mathbf{m_w}} \prod_{i=1}^{\kappa} \left(\log \frac{x_i}{M_i} \right)^{d_i - 1} =$$

$$\frac{\prod_{i=1}^{\kappa} (d_i - 1)!}{(2\pi i)^\kappa} \int_{(b)} \cdots \int_{(b)} \sum_{m_1, \ldots, m_w}^{\infty} \frac{a_{\mathbf{m_w}}}{\prod_{i=1}^{\kappa} M_i^{s_i}} \prod_{i=1}^{\kappa} \frac{x_i^{s_i}}{s_i^{d_i}} ds_1 \cdots ds_\kappa,$$

provided that $\displaystyle\sum_{m_1, \ldots, m_w}^{\infty} \frac{|a_{\mathbf{m_w}}|}{\prod_{i=1}^{\kappa} M_i^b}$ converges, and that for each i, $x_i \neq M_i$ for all possible M_i.

Proof

By the calculus of residue, we have

$$\frac{1}{2\pi i} \int_{(b)} \frac{y^s}{s^d} ds = \begin{cases} \dfrac{(\log y)^{d-1}}{(d-1)!} & \text{if } y > 1; \\ 0 & \text{if } 0 < y < 1. \end{cases}$$

Hence, for all sufficiently large B,

$$\sum_{\substack{m_1,\ldots,m_w=1\\ M_i\le x_i\ (1\le i\le\kappa)}}^{\infty} a_{\mathbf{m_w}}\prod_{i=1}^{\kappa}\left(\log\frac{x_i}{M_i}\right)^{d_i-1} =$$

$$\frac{\prod_{i=1}^{\kappa}(d_i-1)!}{(2\pi i)^{\kappa}}\int_{(b)}\cdots\int_{(b)}\sum_{m_1,\ldots,m_w}^{B}\frac{a_{\mathbf{m_w}}}{\prod_{i=1}^{\kappa}M_i^{s_i}}\prod_{i=1}^{\kappa}\frac{x_i^{s_i}}{s_i^{d_i}}ds_1\cdots ds_{\kappa}.$$

Given $\varepsilon > 0$, by Lemma 2.3.4(ii) and the assumption that $\displaystyle\sum_{m_1,\ldots,m_w}^{\infty}\frac{|a_{\mathbf{m_w}}|}{\prod_{i=1}^{\kappa}M_i^b}$ converges, $\exists B_0$ such that for all $B > B_0$,

$$\left|\frac{\prod_{i=1}^{\kappa}(d_i-1)!}{(2\pi i)^{\kappa}}\int_{(b)}\cdots\int_{(b)}\sum_{\substack{m_1,\ldots,m_w\\ m_i>B\text{ for some }i}}\frac{a_{\mathbf{m_w}}}{\prod_{i=1}^{\kappa}M_i^{s_i}}\prod_{i=1}^{\kappa}\frac{x_i^{s_i}}{s_i^{d_i}}ds_1\cdots ds_{\kappa}\right|$$

$$\le \quad \frac{\prod_{i=1}^{\kappa}(d_i-1)!}{(2\pi)^{\kappa}}\sum_{\substack{m_1,\ldots,m_w\\ m_i>B_0\text{ for some }i}}\frac{|a_{\mathbf{m_w}}|}{\prod_{i=1}^{\kappa}M_i^b}\left(\prod_{i=1}^{\kappa}x_i^b\int_{(b)}\frac{|ds_i|}{|s_i|^{d_i}}\right) < \varepsilon.$$

The result follows.

Q.E.D.

<u>Lemma 2.4.5</u>

For $1 \le i \le \kappa$, let C_i be the circle $|s_i| = d_i$ described in the positive sense. Then

$$\int_{C_\kappa}\int_{C_{\kappa-1}}\cdots\int_{C_1}\Phi(s_1,\ldots,s_{\kappa-1},s_\kappa)ds_1\cdots ds_{\kappa-1}ds_\kappa =$$

$$\int_{C_\kappa}\int_{C'_{\kappa-1}}\cdots\int_{C'_1}\Phi(s_\kappa\xi_1,\ldots,s_\kappa\xi_{\kappa-1},s_\kappa)\cdot s_\kappa^{\kappa-1}d\xi_1\cdots d\xi_{\kappa-1}ds_\kappa,$$

where C'_i is the circle $|\xi_i| = d_i/d_\kappa$ described in the positive sense.

<u>Proof</u>

Observe that

$$\begin{aligned} L.H.S. &= \int_{C_\kappa}\int_0^{2\pi}\cdots\int_0^{2\pi}\Phi\left(\frac{s_\kappa d_1e^{i\theta_1}}{d_\kappa},\ldots,\frac{s_\kappa d_{\kappa-1}e^{i\theta_{\kappa-1}}}{d_\kappa},s_\kappa\right)\times \\ &\qquad \frac{s_\kappa d_1ie^{i\theta_1}}{d_\kappa}\cdots\frac{s_\kappa d_{\kappa-1}ie^{i\theta_{\kappa-1}}}{d_\kappa}d\theta_1\cdots d\theta_{\kappa-1}ds_\kappa \\ &= R.H.S. \end{aligned}$$

Q.E.D.

Lemma 2.4.6

Let $F : [a,b] \times [c,d] \to \mathbb{R}$ be continuous. Suppose f, g are two continuous functions defined on $[c,d]$ such that $a \le f(y) \le g(y) \le b$. Define $G : [c,d] \to \mathbb{R}$ by

$$G(y) = \int_{f(y)}^{g(y)} F(x,y)dx.$$

If $F, f, g \in \mathcal{C}^\infty$, then so is G, and

$$\frac{dG}{dy} = \int_{f(y)}^{g(y)} \frac{\partial F}{\partial y}(x,y)dx + F(g(y),y)g'(y) - F(f(y),y)f'(y).$$

Proof

This can be found in e.g.[20, Chapter 10, section 7.3].

Q.E.D.

Lemma 2.4.7 (Riemann-Stieltjes Integration)

Write "$f \in \mathsf{RS}(\alpha)$ on $[a,b]$" if f is Riemann-integrable with respect to α on $[a,b]$. We have the following propositions.

(a) *If $f \in \mathsf{RS}(\alpha)$ and $f \in \mathsf{RS}(\beta)$ on $[a,b]$, then for all $c_1, c_2 \in \mathbb{R}$, we have $f \in \mathsf{RS}(c_1\alpha + c_2\beta)$ on $[a,b]$ and*

$$\int_a^b f\, d(c_1\alpha + c_2\beta) = c_1 \int_a^b f\, d\alpha + c_2 \int_a^b f\, d\beta.$$

(b) *If $f \in \mathsf{RS}(\alpha)$ on $[a,b]$, then $\alpha \in \mathsf{RS}(f)$ on $[a,b]$ and we have*

$$\int_a^b f\, d\alpha + \int_a^b \alpha\, df = f(b)\alpha(b) - f(a)\alpha(a).$$

(c) *If $f \in \mathsf{RS}(\alpha)$ on $[a,b]$ and α has a continuous derivative on $[a,b]$, then $\int_a^b f(x)\alpha'(x)\, dx$ exists and*

$$\int_a^b f\, d\alpha = \int_a^b f(x)\alpha'(x)\, dx.$$

(d) *Let α be a step function defined on $[a,b]$ with jumps α_k at x_k. Suppose f is a continuous real-valued function defined on $[a,b]$. Then $\int_a^b f\,d\alpha$ exists and*

$$\int\limits_a^b f\,d\alpha = \sum_{k=1}^{n} f(x_k)\alpha_k.$$

Proof

See [1, Theorem 7.3, 7.6, 7.8 and 7.11].

Q.E.D.

Proposition 2.4.8

Let $j \in \mathbb{N}$ and $V_j : \prod_{\ell=1}^{j}[a_\ell, b_\ell] \to \mathbb{R}$ be a function in $\mathcal{C}^\infty$, where $a_\ell \geq 1.5$. For $1 \leq i \leq j-1$, let $f_i, g_i : \prod_{\ell=1}^{i}[a_\ell, b_\ell] \to [a_{i+1}, b_{i+1}]$ be functions in $\mathcal{C}^\infty$ such that $f_i(\mathbf{y_i}) \leq g_i(\mathbf{y_i})$.
Define $\Xi_i : \prod_{\ell=1}^{i}[a_\ell, b_\ell] \to \mathbb{R}$, $V_i : \prod_{\ell=1}^{i}[a_\ell, b_\ell] \to \mathbb{R}$ recursively from $i = j$ to $i = 1$ by:

$$\Xi_j(\mathbf{y_j}) = \frac{V_j(\mathbf{y_j})}{\log y_j}, \qquad V_{j-1}(\mathbf{y_{j-1}}) = \int\limits_{f_{j-1}(\mathbf{y_{j-1}})}^{g_{j-1}(\mathbf{y_{j-1}})} \Xi_j(\mathbf{y_{j-1}}, u_j)du_j,$$

$$\vdots$$

$$\Xi_i(\mathbf{y_i}) = \frac{V_i(\mathbf{y_i})}{\log y_i}, \qquad V_{i-1}(\mathbf{y_{i-1}}) = \int\limits_{f_{i-1}(\mathbf{y_{i-1}})}^{g_{i-1}(\mathbf{y_{i-1}})} \Xi_i(\mathbf{y_{i-1}}, u_i)du_i,$$

$$\vdots$$

$$\Xi_1(\mathbf{y_1}) = \frac{V_1(\mathbf{y_1})}{\log y_1}, \qquad V_0 = \int\limits_{a_1}^{b_1} \Xi_1(u_1)du_1.$$

Let $f_0(\mathbf{p_0}) = a_1$, $g_0(\mathbf{p_0}) = b_1$, and write

$$\begin{aligned} \vartheta(x) &:= \sum_{p<x} \log p = x + \mathfrak{R}(x), \\ {\sum_{f_i(\mathbf{p_i})}^{g_i(\mathbf{p_i})}}' &:= \sum_{f_i(\mathbf{p_i})<p_{i+1}\leq g_i(\mathbf{p_i})} . \end{aligned}$$

Then

$$\sum_{f_0(\mathbf{p_0})}^{g_0(\mathbf{p_0})}{}' \sum_{f_1(\mathbf{p_1})}^{g_1(\mathbf{p_1})}{}' \cdots \sum_{f_{j-1}(\mathbf{p_{j-1}})}^{g_{j-1}(\mathbf{p_{j-1}})}{}' V_j(\mathbf{p_j}) = V_0 + R_1 + \cdots + R_j,$$

where

$$R_i = \sum_{f_0(\mathbf{p_0})}^{g_0(\mathbf{p_0})}{}' \sum_{f_1(\mathbf{p_1})}^{g_1(\mathbf{p_1})}{}' \cdots \sum_{f_{i-2}(\mathbf{p_{i-2}})}^{g_{i-2}(\mathbf{p_{i-2}})}{}' \left\{ \Big[\mathfrak{R}(u_i)\Xi_i(\mathbf{p_{i-1}}, u_i) \Big]_{u_i=f_{i-1}(\mathbf{p_{i-1}})}^{u_i=g_{i-1}(\mathbf{p_{i-1}})} - \int_{f_{i-1}(\mathbf{p_{i-1}})}^{g_{i-1}(\mathbf{p_{i-1}})} \frac{\partial \Xi_i}{\partial y_i}(\mathbf{p_{i-1}}, u_i)\mathfrak{R}(u_i)du_i \right\}.$$

Proof

This is proved by induction on j.

1. When $j = 1$,

$$\begin{aligned} L.H.S. &= \sum_{a_1 < p_1 \leq b_1} V_1(p_1) \\ &= \int_{a_1}^{b_1} \frac{V_1(u_1)}{\log u_1} d\vartheta(u_1) && \text{by Lemma 2.4.7(d)} \\ &= \int_{a_1}^{b_1} \frac{V_1(u_1)}{\log u_1} du_1 + \int_{a_1}^{b_1} \frac{V_1(u_1)}{\log u_1} d\mathfrak{R}(u_1) && \text{by Lemma 2.4.7(a),} \end{aligned}$$

   since the integrand is in both $\mathsf{RS}(\vartheta(u_1))$ and $\mathsf{RS}(u_1)$.

   The first term is V_0. The second term, by Lemma 2.4.7(b) and (c), equals

$$\Big[\mathfrak{R}(u_1)\Xi_1(u_1) \Big]_{u_1=f_0(\mathbf{p_0})}^{u_1=g_0(\mathbf{p_0})} - \int_{f_0(\mathbf{p_0})}^{g_0(\mathbf{p_0})} \Xi_1'(u_1)\mathfrak{R}(u_1)du_1 = R_1.$$

   It follows that $L.H.S. = V_0 + R_1 = R.H.S.$

2. Suppose the statement is true for $j = r$. When $j = r + 1$,

$$L.H.S. = \sum_{f_0(\mathbf{p_0})}^{g_0(\mathbf{p_0})}{}' \sum_{f_1(\mathbf{p_1})}^{g_1(\mathbf{p_1})}{}' \cdots \sum_{f_r(\mathbf{p_r})}^{g_r(\mathbf{p_r})}{}' V_{r+1}(\mathbf{p_{r+1}}).$$

By the argument in 1., this is

$$= \sum_{f_0(\mathbf{p_0})}^{g_0(\mathbf{p_0})}{}' \left(\sum_{f_1(\mathbf{p_1})}^{g_1(\mathbf{p_1})}{}' \cdots \left(\sum_{f_{r-1}(\mathbf{p_{r-1}})}^{g_{r-1}(\mathbf{p_{r-1}})}{}' \right.\right.$$

$$\left\{ \int_{f_r(\mathbf{p_r})}^{g_r(\mathbf{p_r})} \frac{V_{r+1}(\mathbf{p_r}, u_{r+1})}{\log u_{r+1}} du_{r+1} + \Big[\mathfrak{R}(u_{r+1}) \Xi_{r+1}(\mathbf{p_r}, u_{r+1}) \Big]_{u_{r+1}=f_r(\mathbf{p_r})}^{u_{r+1}=g_r(\mathbf{p_r})} \right.$$

$$\left.\left.\left. - \int_{f_r(\mathbf{p_r})}^{g_r(\mathbf{p_r})} \frac{\partial \Xi_{r+1}}{\partial y_{r+1}} (\mathbf{p_r}, u_{r+1}) \mathfrak{R}(u_{r+1}) du_{r+1} \right\} \right) \right)$$

$$= \sum_{f_0(\mathbf{p_0})}^{g_0(\mathbf{p_0})}{}' \left(\sum_{f_1(\mathbf{p_1})}^{g_1(\mathbf{p_1})}{}' \cdots \left(\sum_{f_{r-1}(\mathbf{p_{r-1}})}^{g_{r-1}(\mathbf{p_{r-1}})}{}' V_r(\mathbf{p_r}) \right) \right) + R_{r+1}.$$

Now $V_r : \prod_{\ell=1}^{r} [a_\ell, b_\ell] \to \mathbb{R}$ given by

$$V_r(\mathbf{y_r}) = \int_{f_r(\mathbf{y_r})}^{g_r(\mathbf{y_r})} \Xi_{r+1}(\mathbf{y_r}, u_{r+1}) du_{r+1}$$

is in $\mathcal{C}^\infty$ by Lemma 2.4.6. Therefore, we can apply the induction hypothesis to conclude that $L.H.S. = V_0 + R_1 + \cdots + R_{r+1} = R.H.S.$

Q.E.D.

Proposition 2.4.9

Let $N^\varepsilon < Y < N^{2\varepsilon}$. Suppose $f_i, g_i : \prod_{\ell=1}^{i} [a_\ell, b_\ell] \to [a_{i+1}, b_{i+1}]$ $(0 \le i \le j-1)$ are $\mathcal{C}^\infty$-functions satisfying the following properties:

(i) $Y < f_i(\mathbf{y_i}) \le g_i(\mathbf{y_i}) \le N$.

(ii) *For all $y_1, \ldots, y_j$ such that $f_0 < y_1 \le g_0$, $\ldots$, $f_{j-1}(\mathbf{y_{j-1}}) < y_j \le g_{j-1}(\mathbf{y_{j-1}})$, we have* $\dfrac{N}{y_1 \cdots y_j} > N^{\varepsilon'}$.

(iii) $\dfrac{\partial h_l / \partial y_i}{h_l}(\mathbf{u_l}) \ll \dfrac{1}{u_i}$ *for any $1 \le i \le l \le j-1$, where h_l is f_l or g_l, and the implied constant only depends on h_l.*

Then when we apply Proposition 2.4.8 to

$$V_j(\mathbf{y_j}) = \frac{(\log y_1)^{m_1}\cdots(\log y_j)^{m_j}}{y_1\cdots y_j \log\dfrac{N}{y_1\cdots y_j}},$$

where $m_1 \ldots m_j$ are non-negative integers, all the R_i that occur are $\ll \dfrac{1}{\log^{\mathsf{A}} N}$, where A can be any positive number. The implied constant only depends on A, ε, ε', j, f_i, g_i and m_i.

Proof

By induction on i from $i = j$ to $i = 1$, we see that

$$\Xi_i(\mathbf{y_i}) = \int_{f_i(\mathbf{y_i})}^{g_i(\mathbf{y_i})}\cdots\int_{f_{j-1}(\mathbf{y_i},u_{i+1},\ldots,u_{j-1})}^{g_{j-1}(\mathbf{y_i},u_{i+1},\ldots,u_{j-1})}\left(\log\frac{N}{y_1\cdots y_i u_{i+1}\cdots u_j}\right)^{-1}$$
$$\frac{(\log y_1)^{m_1}\cdots(\log y_i)^{m_i}(\log u_{i+1})^{m_{i+1}}\cdots(\log u_j)^{m_j}}{y_1\cdots y_{i-1}(y_i\log y_i)(u_{i+1}\log u_{i+1})\cdots(u_j\log u_j)}du_j\cdots du_{i+1}.$$

Observe that if $Y < y_i \le N$, then

$$\mathfrak{R}(y_i)\frac{(\log y_i)^{m_i}}{(y_i\log y_i)} \ll \frac{y_i}{\log^{\mathbb{A}} y_i}\frac{(\log y_i)^{m_i}}{(y_i\log y_i)} \ll \frac{1}{\log^{\mathbb{A}} N}.$$

Here we have used the prime number theorem: For any $\mathsf{A} > 0$, we have

$$\mathfrak{R}(u) \ll \frac{u}{\log^{\mathsf{A}} u},$$

where the implied constant only depends on A.

Hence, by property(i) and (ii), we have

$$\sideset{}{'}\sum_{f_0(\mathbf{p_0})}^{g_0(\mathbf{p_0})}\sideset{}{'}\sum_{f_1(\mathbf{p_1})}^{g_1(\mathbf{p_1})}\cdots\sideset{}{'}\sum_{f_{i-2}(\mathbf{p_{i-2}})}^{g_{i-2}(\mathbf{p_{i-2}})}\Big[\mathfrak{R}(u_i)\Xi_i(\mathbf{p_{i-1}},u_i)\Big]_{u_i=f_{i-1}(\mathbf{p_{i-1}})}^{u_i=g_{i-1}(\mathbf{p_{i-1}})}$$
$$\ll \sum_{Y<p_1\le N}\sum_{Y<p_2\le N}\cdots\sum_{Y<p_{i-1}\le N}\frac{(\log p_1)^{m_1}\cdots(\log p_{i-1})^{m_{i-1}}}{p_1\cdots p_{i-1}}\frac{1}{\log^{\mathbb{A}} N}$$
$$\times\int_Y^N\cdots\int_Y^N\frac{(\log u_{i+1})^{m_{i+1}}\cdots(\log u_j)^{m_j}}{(u_{i+1}\log u_{i+1})\cdots(u_j\log u_j)}du_j\cdots du_{i+1}$$
$$\ll \frac{1}{\log^{\mathbb{A}} N}.$$

Therefore, if we can show that

$$\frac{\partial\Xi_i}{\partial y_i}(\mathbf{p_{i-1}},u_i) \ll \frac{(\log p_1)^{m_1}\cdots(\log p_{i-1})^{m_{i-1}}}{p_1\cdots p_{i-1}}\frac{\log^{\mathbb{A}} N}{u_i^2}, \tag{2.2}$$

then

$$\sum_{f_0(\mathbf{p_0})}^{g_0(\mathbf{p_0})}{}' \sum_{f_1(\mathbf{p_1})}^{g_1(\mathbf{p_1})}{}' \cdots \sum_{f_{i-2}(\mathbf{p_{i-2}})}^{g_{i-2}(\mathbf{p_{i-2}})}{}' \int\limits_{f_{i-1}(\mathbf{p_{i-1}})}^{g_{i-1}(\mathbf{p_{i-1}})} \frac{\partial \Xi_i}{\partial y_i}(\mathbf{p_{i-1}}, u_i)\mathfrak{R}(u_i)du_i$$

$$\ll \sum_{Y<p_1\le N} \sum_{Y<p_2\le N} \cdots \sum_{Y<p_{i-1}\le N} \frac{(\log p_1)^{m_1}\cdots(\log p_{i-1})^{m_{i-1}}}{p_1\cdots p_{i-1}} \int\limits_{Y}^{N} \frac{1}{u_i \log^{\mathbb{A}} N} du_i$$

$$\ll \frac{1}{\log^{\mathbb{A}} N},$$

and the proof is complete.

By Lemma 2.4.6, $\dfrac{\partial \Xi_i}{\partial y_i}(\mathbf{p_{i-1}}, u_i)$ is a sum of the term

$$\frac{(\log p_1)^{m_1}\cdots(\log p_{i-1})^{m_{i-1}}}{p_1\cdots p_{i-1}} \int\limits_{f_i(\mathbf{p_{i-1}},u_i)}^{g_i(\mathbf{p_{i-1}},u_i)} \cdots \int\limits_{f_{j-1}(\mathbf{p_{i-1}},u_i,\ldots,u_{j-1})}^{g_{j-1}(\mathbf{p_{i-1}},u_i,\ldots,u_{j-1})} \frac{(\log u_{i+1})^{m_{i+1}}\cdots(\log u_j)^{m_j}}{(u_{i+1}\log u_{i+1})\cdots(u_j \log u_j)}$$
$$\frac{\partial}{\partial y_i}\left[\frac{(\log y_i)^{m_i}}{(y_i \log y_i)}\left(\log\frac{N}{p_1\cdots p_{i-1} y_i u_{i+1}\cdots u_j}\right)^{-1}\right]\Bigg|_{y_i=u_i} du_j\cdots du_{i+1},$$

and terms of the form

$$\pm\frac{(\log p_1)^{m_1}\cdots(\log p_{i-1})^{m_{i-1}}}{p_1\cdots p_{i-1}} \int\limits_{f_i(\mathbf{p_{i-1}},u_i)}^{g_i(\mathbf{p_{i-1}},u_i)} \cdots \int\limits_{f_{l-1}(\mathbf{p_{i-1}},u_i,\ldots,u_{l-1})}^{g_{l-1}(\mathbf{p_{i-1}},u_i,\ldots,u_{l-1})} \int\limits_{f_{l+1}(\mathbf{p_{i-1}},u_i,\ldots,u_{l+1})}^{g_{l+1}(\mathbf{p_{i-1}},u_i,\ldots,u_{l+1})}$$
$$\cdots \int\limits_{f_{j-1}(\mathbf{p_{i-1}},u_i,\ldots,u_{j-1})}^{g_{j-1}(\mathbf{p_{i-1}},u_i,\ldots,u_{j-1})} \left(\log\frac{N}{p_1\cdots p_{i-1}u_i\cdots u_j}\right)^{-1} \frac{(\log u_i)^{m_i}\cdots(\log u_j)^{m_j}}{(u_i \log u_i)\cdots(u_j \log u_j)}$$
$$\frac{\partial h_l}{\partial y_i}(\mathbf{p_{i-1}}, u_i, \ldots, u_l)du_j\cdots du_{l+2}du_l\cdots du_{i+1},$$

where u_{l+1} equals $f_l(\mathbf{p_{i-1}}, u_i, \ldots, u_l)$ or $g_l(\mathbf{p_{i-1}}, u_i, \ldots, u_l)$, and $h_l(\mathbf{y_l})$ equals $f_l(\mathbf{y_l})$ or $g_l(\mathbf{y_l})$ accordingly.

To deal with the first term, note that for any $M > 0$ and any non-negative integer m, we have

$$\left(\frac{(\log u)^m}{u \log u \log\dfrac{M}{u}}\right)' = \frac{(\log u \log\frac{M}{u})m(\log u)^{m-1} - (\log u)^m(\log u \log\frac{M}{u} + \log\frac{M}{u} - \log u)}{u^2 \log^2 u \left(\log\dfrac{M}{u}\right)^2}.$$

Provided that $Y < u \le N$ and $\log N > \log\dfrac{M}{u} \gg \log N$, this is $\ll \dfrac{\log^{\mathbb{A}} N}{u^2}$, where the implied constant only depends on m and M.

Hence, by property(ii), the aforementioned first term is, after dropping the factor $\dfrac{(\log p_1)^{m_1}\cdots(\log p_{i-1})^{m_{i-1}}}{p_1\cdots p_{i-1}}$,

$$\ll \int_Y^N \cdots \int_Y^N \frac{(\log u_{i+1})^{m_{i+1}}\cdots(\log u_j)^{m_j}}{(u_{i+1}\log u_{i+1})\cdots(u_j\log u_j)}\frac{\log^{\mathbb{A}} N}{u_i^2}du_j\cdots du_{i+1} \ll \frac{\log^{\mathbb{A}} N}{u_i^2}.$$

To deal with the remaining terms, we make use of property(iii) . After dropping the leading factor as above, each of them is

$$\ll \int_Y^N \cdots \int_Y^N \frac{\log^{\mathbb{A}} N \cdot \dfrac{\partial h_l}{\partial y_i}(\mathbf{p_{i-1}}, u_i, \ldots, u_l)}{u_i\, h_l(\mathbf{p_{i-1}}, u_i, \ldots, u_l)} \frac{du_j\cdots du_{l+2}du_l\cdots du_{i+1}}{u_{i+1}\cdots u_l u_{l+2}\cdots u_j} \ll \frac{\log^{\mathbb{A}} N}{u_i^2}.$$

This shows (2.2). Done.

Q.E.D.

2.5 E_j-numbers in arithmetic progressions

Recall Definition 2.2.1. Throughout this section, we assume x is a large real number, and our aim is to prove the following result:

Proposition 2.5.1

Suppose $x^{\epsilon} < y < x^{\frac{1}{2}}$, $y^2 < t \le x$, $a \in \mathbb{N}$. *For any* $\mathsf{A} > 1$, $h \in \mathbb{N}$, *there exists* $\mathsf{B} = \mathsf{B}(\mathsf{A}, h)$ *such that*

$$\sum_{q \le \frac{\sqrt{x}}{a \log^{\mathsf{B}}(x)}} \mu^2(q) h^{\omega(q)} \mathcal{E}_E^*(x; aq) \ll \frac{x}{\log^{\mathsf{A}} x},$$

where E *can be any one of the following:*

(a) $E(m) = \begin{cases} 1 & \textit{if } m = p_1p_2,\ y < p_1 \le t^{\frac{1}{2}} < p_2 \\ 0 & \textit{otherwise;} \end{cases}$

(b) $E(m) = \begin{cases} 1 & \textit{if } m = p_1\cdots p_j,\ y < p_1 < p_2 < \cdots < p_j \\ 0 & \textit{otherwise,} \end{cases}$

where j *is any integer* ≥ 2;

(c) $E(m) = \varpi(m)$.

The implied constant only depends on A, h, *and* ϵ *for* (a) *and* (b).

We begin by several lemmas.

Lemma 2.5.2

Let $M = x^\theta$, where $0 < \epsilon \le \theta \le 1-\epsilon$, and let $0 < c < \frac{1}{2}$. Define

$$\beta_m = \begin{cases} 1 & \text{if } M < m \le \frac{2M}{1+c},\ m \text{ is a prime and } x^d < m \le x^{d'}, \\ 0 & \text{otherwise,} \end{cases}$$

where $d' > d \ge 0$. Then for any $e \ge 1$, $f > 1$, $(a,f) = 1$, $\mathsf{A} > 1$, we have

$$\left| \sum_{\substack{m \equiv a (\bmod f) \\ (m,e)=1}} \beta_m - \frac{1}{\phi(f)} \sum_{(m,ef)=1} \beta_m \right| \ll \frac{M\tau(e)}{\log^{\mathsf{A}} M} \tag{2.3}$$

where the implied constant depends on A alone and so is uniform in θ, c, d, d', f.

Proof

If $\frac{2M}{1+c} < x^d$, then the *L.H.S.* of (2.3) is zero. Let's assume $x^d \le \frac{2M}{1+c}$.

When $f \ge \log^{2\mathsf{A}} M$,

$$\begin{aligned} L.H.S. &\le \sum_{\substack{m \equiv a (\bmod f) \\ M < m \le \frac{2M}{1+c}}} \varpi(m) + \frac{1}{\phi(f)} \sum_{M < m \le \frac{2M}{1+c}} \varpi(m) \\ &\ll \frac{M}{f} + \frac{M}{\phi(f)} + 1 \ll \frac{M}{f} + \frac{M \log f}{f} + 1 \ll \frac{M}{\log^{\mathsf{A}} M}. \end{aligned}$$

On the other hand, when $f < \log^{2\mathsf{A}} M$,

$$\begin{aligned} L.H.S. &\le \left| \sum_{m \equiv a (\bmod f)} \beta_m - \frac{1}{\phi(f)} \sum_m \beta_m \right| + 2\tau(e) + \frac{\tau(f)}{\phi(f)} \\ &\ll \left| \sum_{\substack{m \equiv a (\bmod f) \\ m \le \min(\frac{2M}{1+c}, x^{d'})}} \varpi(m) - \frac{1}{\phi(f)} \sum_{m \le \min(\frac{2M}{1+c}, x^{d'})} \varpi(m) \right| \\ &\quad + \left| \sum_{\substack{m \equiv a (\bmod f) \\ m \le \max(M, x^d)}} \varpi(m) - \frac{1}{\phi(f)} \sum_{m \le \max(M, x^d)} \varpi(m) \right| + 2\tau(e) + \frac{\sqrt{f} \log f}{f}. \end{aligned}$$

By Siegel-Walfisz theorem[4, Chapter 22] and $x^d \le \frac{2M}{1+c}$, this is $\ll \dfrac{M\tau(e)}{\log^{\mathsf{A}} M}$.

Q.E.D.

Lemma 2.5.3 (c.f. [2, Theorem 0])

Let θ, ϵ, c, β_m be that in Lemma 2.5.2. If α_m is supported in $[x^{1-\theta}, (1+c)x^{1-\theta}]$, $|\alpha_m| \le 1$, then for any $\mathsf{A} > 1$, there exists B, which only depends on A, such that

$$\sum_{q \le \frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}} \mathcal{E}^*_{\alpha*\beta}(x;q) \ll \frac{x}{\log^{\mathsf{A}} x},$$

where

$$\alpha * \beta(m) := \sum_{d|m} \alpha_d \beta_{m/d}.$$

The implied constant only depends on A and ϵ.

Proof

Since

$$\begin{aligned}
\mathcal{E}_{\alpha*\beta}(x;q,a) &= \sum_{x<\ell\le 2x} \frac{\alpha*\beta(\ell)}{\phi(q)} \sum_{\chi\in X_q} \chi(\ell)\bar{\chi}(a) - \frac{1}{\phi(q)} \sum_{\substack{x<\ell\le 2x \\ (\ell,q)=1}} \alpha*\beta(\ell) \\
&= \sum_{\substack{\chi\in X_q \\ \chi\neq\chi_0}} \bar{\chi}(a) \sum_{x<\ell\le 2x} \frac{\alpha*\beta(\ell)\chi(\ell)}{\phi(q)} \\
&= \frac{1}{\phi(q)} \sum_{\substack{\chi\in X_q \\ \chi\neq\chi_0}} \bar{\chi}(a) \left(\sum_n \beta_n \chi(n)\right)\left(\sum_m \alpha_m \chi(m)\right),
\end{aligned}$$

we have

$$\sum_{q \le \frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}} \max_{(a,q)=1} |\mathcal{E}_{\alpha*\beta}(x;q,a)| \le \sum_{q \le \frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}} \frac{1}{\phi(q)} \sum_{\substack{\chi\in X_q \\ \chi\neq\chi_0}} \left|\sum_n \beta_n\chi(n)\right| \left|\sum_m \alpha_m \chi(m)\right|.$$

Let χ be induced by $\psi \mod f$, where $q = fe, f > 1$. Since $\phi(fe) \ge \phi(f)\phi(e)$, the right hand side is

$$\begin{aligned}
&\le \sum_{e \le \frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}} \frac{1}{\phi(e)} \sum_{2\le f \le \frac{\sqrt{x}}{e\log^{\mathsf{B}}(x)}} \frac{1}{\phi(f)} \sum_{\psi\in X_f^*} \left| \sum_{\substack{n \\ (n,e)=1}} \beta_n \psi(n)\right| \left|\sum_{\substack{m \\ (m,e)=1}} \alpha_m \psi(m)\right| \\
&:= \sum_{e \le \frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}} \frac{1}{\phi(e)} (T_e(f\le F) + T_e(f>F)), \text{ where } F := \log^{\mathsf{A}+4} x.
\end{aligned}$$

By Cauchy's inequality,

$$\begin{aligned} T_e(f\le F) \ \le\ & \left\{\sum_{2\le f\le F}\frac{1}{\phi(f)}\sum_{\psi\in X_f^*}\left|\sum_{\substack{n\\(n,e)=1}}\beta_n\psi(n)\right|^2\right\}^{1/2}\times \\ & \left\{\sum_{2\le f\le F}\frac{1}{\phi(f)}\sum_{\psi\in X_f^*}\left|\sum_{\substack{m\\(m,e)=1}}\alpha_m\psi(m)\right|^2\right\}^{1/2}. \end{aligned}$$

Since

$$\begin{aligned} \sum_{\substack{n\\(n,e)=1}}\beta_n\psi(n) &= \sum_{a\in(\mathbb{Z}/f\mathbb{Z})^*}\ \sum_{\substack{n\equiv a(\text{mod } f)\\(n,e)=1}}\beta_n\psi(a) \\ &= \sum_{a\in(\mathbb{Z}/f\mathbb{Z})^*}\ \sum_{\substack{n\equiv a(\text{mod } f)\\(n,e)=1}}\beta_n\psi(a) - \sum_{a\in(\mathbb{Z}/f\mathbb{Z})^*}\frac{\psi(a)}{\phi(f)}\sum_{(n,ef)=1}\beta_n, \end{aligned}$$

so by Lemma 2.5.2, this is $\ll \dfrac{fM\tau(e)}{\log^{2.5\mathsf{A}+9}M}$. By the large sieve inequality (e.g. [4, Chapter 27 Theorem 4]), it follows that

$$\begin{aligned} T_e(f\le F) &\ll \left\{\sum_{2\le f\le F}\frac{f^2M^2\tau^2(e)}{\log^{5\mathsf{A}+18}M}\right\}^{1/2}\left\{(F^2+x^{1-\theta})\sum|\alpha_m|^2\right\}^{1/2} \\ &\ll \frac{F^{3/2}M\tau(e)}{\log^{2.5\mathsf{A}+9}M}\cdot x^{1-\theta}\ll\frac{x\tau(e)}{\log^{\mathsf{A}+3}x}. \end{aligned}$$

This gives

$$\sum_{e\le\frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}}\frac{1}{\phi(e)}T_e(f\le F)\ll\frac{x}{\log^{\mathsf{A}+3}x}\sum_{e\le\frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}}\tau(e)\frac{\log x}{e}\ll\frac{x\log^3x}{\log^{\mathsf{A}+3}x}\ll\frac{x}{\log^{\mathsf{A}}x}.$$

For $T_e(f>F)$, we split the sum over f into $\ll \log x$ intervals of type (V,2V], and we apply Cauchy's inequality to each of these. In this way,

$$\begin{aligned} T_e(f>F) \ \ll\ & \sum_V\left\{\sum_{V\le f\le 2V}\frac{1}{\phi(f)}\sum_{\psi\in X_f^*}\left|\sum_{\substack{n\\(n,e)=1}}\beta_n\psi(n)\right|^2\right\}^{1/2}\times \\ & \left\{\sum_{V\le f\le 2V}\frac{1}{\phi(f)}\sum_{\psi\in X_f^*}\left|\sum_{\substack{m\\(m,e)=1}}\alpha_m\psi(m)\right|^2\right\}^{1/2}. \end{aligned}$$

By the large sieve inequality, this is

$$
\begin{aligned}
&\ll \sum_{V} \frac{1}{V}\left\{(V^2+M)\sum|\beta_n|^2\right\}^{1/2}\left\{(V^2+x^{1-\theta})\sum|\alpha_m|^2\right\}^{1/2}\\
&\ll (\log x)\max_{F<V\le\frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}} \frac{1}{V}\left\{(V^2+M)\sum|\beta_n|^2\right\}^{1/2}\left\{(V^2+x^{1-\theta})\sum|\alpha_m|^2\right\}^{1/2}\\
&\ll x^{\frac{1}{2}}(\log x)\max_{F<V\le\frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}} \frac{1}{V}(V+\sqrt{M})(V+x^{\frac{1-\theta}{2}})\\
&\ll x^{\frac{1}{2}}(\log x)\left(\frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}+x^{\frac{1-\theta}{2}}+\sqrt{M}+x^{\frac{1}{2}}F^{-1}\right).
\end{aligned}
$$

By choosing $\mathsf{B}=\mathsf{A}+3$, this is $\ll \dfrac{x}{\log^{\mathsf{A}+2}x}$, and so

$$
\sum_{e\le\frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}}\frac{1}{\phi(e)}T_e(f>F)\ll\frac{x}{\log^{\mathsf{A}+2}x}\sum_{e\le\frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}}\frac{\log x}{e}\ll\frac{x\log^2 x}{\log^{\mathsf{A}+2}x}\ll\frac{x}{\log^{\mathsf{A}}x}.
$$

Q.E.D.

<u>**Lemma 2.5.4**</u>

Let $j\ge 2$ be a fixed integer. Let (d_r,d'_r) $(r=1,\dots,j)$ be such that

(a) $d_r<d'_r$*;*

(b) $(d_r,d'_r)=(d_{r+1},d'_{r+1})$ *or* $d'_r\le d_{r+1}$*; and*

(c) $d_1>\epsilon$, $d'_j<1-\epsilon$.

Define

$$
w_j(m)=\begin{cases}1 & \text{if } m=p_1\cdots p_j,\ x^{d_r}<p_r\le x^{d'_r} \text{ for } 1\le r\le j,\\ 0 & \text{otherwise.}\end{cases}
$$

Then for any $\mathsf{A}>1$, there exists B, which only depends on A, so is independent of j, d_r and d'_r, such that

$$
\sum_{q\le\frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}}\mathcal{E}^*_{w_j}(x;q)\ll\frac{x}{\log^{\mathsf{A}}x}.
$$

The implied constant only depends on A and ϵ.

Proof (c.f. [19, proof of Lemma 3.2])

By assumption (a) and (b), there exists r_0 such that $(d_r, d_r') = (d_1, d_1')$ if and only if $1 \le r \le r_0$. Let $0 < c < 1/2$. We split the interval $[x^\epsilon, 2x^{1-\epsilon}]$ into $\ll \frac{\log x}{c}$ intervals of type $[t, (1+c)t)$. Given such t, let

$$\alpha_t(m) = \begin{cases} r_0^{-1} & \text{if } m \in [t,(1+c)t),\ m = p_2\cdots p_j,\ x^{d_r} < p_r \le x^{d_r'}\ (2 \le r \le j) \\ 0 & \text{otherwise;} \end{cases}$$

$$\beta_t(m) = \begin{cases} 1 & \text{if } m \in (\frac{x}{t}, \frac{2x}{(1+c)t}],\ m \text{ is a prime and } x^{d_1} < m \le x^{d_1'} \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 2.5.3, there exists B, only depending on A, such that uniformly in t,

$$\sum_{q \le \frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}} \mathcal{E}^*_{\alpha_t * \beta_t}(x;q) \ll \frac{x}{\log^{2\mathsf{A}+3} x}.$$

Define $h(m) := \sum_t \alpha_t * \beta_t(m)$. We have

$$\sum_{q \le \frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}} \mathcal{E}^*_h(x;q) \ll \frac{\log x}{c} \frac{x}{\log^{2\mathsf{A}+3} x}.$$

For $m \in [x, 2x]$, $\displaystyle\sum_t \alpha_t * \beta_t(m) = \sum_{p|m} \sum_t \alpha_t(m/p)\beta_t(p)$

$$= \begin{cases} \displaystyle\sum_{i=1}^{r_0} \sum_t \alpha_t(m/p_i)\beta_t(p_i) & \text{if } m = p_1 \cdots p_j,\ x^{d_r} < p_r \le x^{d_r'} \text{ for } 1 \le r \le j \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \displaystyle r_0^{-1} \sum_{i=1}^{r_0} \beta_{t_i}(p_i) & \begin{array}{l} \text{if } m = p_1 \cdots p_j,\ x^{d_r} < p_r \le x^{d_r'} \text{ for } 1 \le r \le j, \\ \text{and } m/p_i \in [t_i, (1+c)t_i) \end{array} \\ 0 & \text{otherwise.} \end{cases}$$

Here we note that by assumption(c) such t_i must exist. If in addition we restrict $m \in [(1+c)x, \frac{2x}{1+c}]$, then

$$m/p_i \in [t_i, (1+c)t_i) \Rightarrow \frac{m}{(1+c)t_i} < p_i \le \frac{m}{t_i} \Rightarrow \frac{x}{t_i} < p_i \le \frac{2x}{(1+c)t_i},$$

so $h(m) = w_j(m)$ for $m \in [(1+c)x, \frac{2x}{1+c}]$.

Therefore,

$$\begin{aligned}
&\sum_{q\le\frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}} \mathcal{E}^*_{w_j}(x;q)\\
\le\ & \sum_{q\le\frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}} \mathcal{E}^*_h(x;q) + \sum_{q\le\frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}} \max_{(a,q)=1}\left|\sum_{\substack{x<m\le(1+c)x\\ \frac{2x}{1+c}<m\le 2x\\ m\equiv a(\mathrm{mod}\ q)}} w_j(m) - \frac{1}{\phi(q)}\sum_{\substack{x<m\le(1+c)x\\ \frac{2x}{1+c}<m\le 2x\\ (m,q)=1}} w_j(m)\right|\\
&+\sum_{q\le\frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}} \max_{(a,q)=1}\left|\sum_{\substack{x<m\le(1+c)x\\ \frac{2x}{1+c}<m\le 2x\\ m\equiv a(\mathrm{mod}\ q)}} h(m) - \frac{1}{\phi(q)}\sum_{\substack{x<m\le(1+c)x\\ \frac{2x}{1+c}<m\le 2x\\ (m,q)=1}} h(m)\right|\\
\ll\ & \frac{x}{c\log^{2\mathsf{A}+2}x} + \sum_{q\le\frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}} \frac{cx}{\phi(q)}\\
\ll\ & \frac{x}{c\log^{2\mathsf{A}+2}x} + \sum_{q\le\frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}} \frac{cx\log x}{q} \ll \frac{x}{c\log^{2\mathsf{A}+2}x} + cx\log^2 x.
\end{aligned}$$

Taking $c = \log^{-\mathsf{A}-2}x$, the result follows.

Q.E.D.

Lemma 2.5.5 (c.f. [13, Theorem 2])

Suppose $x^{\epsilon}<y<x^{\frac{1}{2}}$, $y^2<t\le x$. For any $\mathsf{A}>1$, there exists $\mathsf{B}=\mathsf{B}(\mathsf{A})$ such that

$$\sum_{q\le\frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}} \mathcal{E}^*_E(x;q) \ll \frac{x}{\log^{\mathsf{A}}x},$$

where E can be any one of the following:

(a) $E(m) = \begin{cases} 1 & \textit{if } m = p_1p_2,\ y<p_1\le t^{\frac{1}{2}}<p_2 \\ 0 & \textit{otherwise;} \end{cases}$

(b) $E(m) = \begin{cases} 1 & \textit{if } m = p_1\cdots p_j,\ y<p_1<p_2<\cdots<p_j \\ 0 & \textit{otherwise,} \end{cases}$

where j is any integer ≥ 2.

The implied constant only depends on A and ϵ.

Proof

By Lemma 2.5.4, there exists B, which only depends on A, such that

$$\sum_{q\leq\frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}}\mathcal{E}^*_{w_j}(x;q)\ll\frac{x}{\log^{\mathsf{A}}x}.$$

(a) This follows immediately by taking $j=2$, $(d_1,d_1')=(\frac{\log y}{\log x},\frac{1}{2}\frac{\log t}{\log x})$ and $(d_2,d_2')=(\frac{1}{2}\frac{\log t}{\log x},\frac{\log(2x/y)}{\log x})$.

(b) Take $(d_r,d_r')=(\frac{\log y}{\log x},\frac{\log(2x/y)}{\log x})$ for all $1\leq r\leq j$. We have

$$\sum_{q\leq\frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}}\mathcal{E}^*_{h}(x;q)\ll\frac{x}{\log^{\mathsf{A}}x},$$

where

$$h(m):=\begin{cases}1 & \text{if } m=p_1\cdots p_j,\ y<p_1\leq p_2\leq\cdots\leq p_j\\ 0 & \text{otherwise.}\end{cases}$$

Now

$$\begin{aligned}&\sum_{q\leq\frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}}\max_{(a,q)=1}|\mathcal{E}_{h-E}(x;q,a)|\\
=&\sum_{q\leq\frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}}\max_{(a,q)=1}\left|\sum_{\substack{x<m\leq 2x\\ m\equiv a(\bmod q)\\ m\text{ not square free}}}h(m)-\frac{1}{\phi(q)}\sum_{\substack{x<m\leq 2x\\ (m,q)=1\\ m\text{ not square free}}}h(m)\right|\\
\leq&\sum_{q\leq\frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}}\max_{(a,q)=1}\sum_{y<n\leq\sqrt{\frac{2x}{y}}}\sum_{\substack{\frac{x}{n^2}<m\leq\frac{2x}{n^2}\\ mn^2\equiv a(\bmod q)}}1+\sum_{q\leq\frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}}\frac{1}{\phi(q)}\sum_{y<n\leq\sqrt{\frac{2x}{y}}}\sum_{\frac{x}{n^2}<m\leq\frac{2x}{n^2}}1.\end{aligned}$$

Note that if $(a,q)=1$ and $mn^2\equiv a(\bmod q)$, then $(n^2,q)=1$. So the above is

$$\ll\sum_{q\leq\frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}}\sum_{y<n\leq\sqrt{\frac{2x}{y}}}\frac{x}{n^2q}+\sum_{q\leq\frac{\sqrt{x}}{\log^{\mathsf{B}}(x)}}\frac{1}{\phi(q)}\sum_{y<n\leq\sqrt{\frac{2x}{y}}}\frac{x}{n^2}\ll\frac{x\log^2x}{y}\ll\frac{x}{\log^{\mathsf{A}}x}.$$

This completes the proof.

Q.E.D.

We end this chapter by proving Proposition 2.5.1. (c.f. [8, Lemma 3.5])

For E being any one of the three functions concerned, and for $q \le \sqrt{x}$, we have $\mathcal{E}_E^*(x;aq) \ll \frac{x}{\phi(aq)} \le \frac{x}{\phi(q)} \ll \frac{x\log x}{q}$, so

$$\begin{aligned}
&\sum_{q\le\frac{\sqrt{x}}{a\log^{\mathsf{B}}(x)}} \mu^2(q)h^{\omega(q)}\mathcal{E}_E^*(x;aq) \\
\ll\;& \sum_{q\le\frac{\sqrt{x}}{a\log^{\mathsf{B}}(x)}} \mu^2(q)h^{\omega(q)}\sqrt{\frac{x\log x}{q}}\sqrt{\mathcal{E}_E^*(x;aq)} \\
\ll\;& \sqrt{x\log x}\left(\sum_{q\le\frac{\sqrt{x}}{a\log^{\mathsf{B}}(x)}} \frac{\mu^2(q)h^{2\omega(q)}}{q}\right)^{1/2}\left(\sum_{q\le\frac{\sqrt{x}}{a\log^{\mathsf{B}}(x)}} \mathcal{E}_E^*(x;aq)\right)^{1/2},
\end{aligned}$$

by Cauchy's inequality. On the one hand, we apply Lemma 2.3.2 to obtain

$$\left(\sum_{q\le\frac{\sqrt{x}}{a\log^{\mathsf{B}}(x)}} \frac{\mu^2(q)h^{2\omega(q)}}{q}\right)^{1/2} \ll (\log x)^{h^2/2}.$$

On the other hand, by Lemma 2.5.5 and the classical Bombieri-Vinogradov theorem, we can choose a B, which only depends on A and h, such that

$$\left(\sum_{q\le\frac{\sqrt{x}}{a\log^{\mathsf{B}}(x)}} \mathcal{E}_E^*(x;aq)\right)^{1/2} \ll \left(\frac{x}{\log^{2\mathsf{A}+h^2+1}x}\right)^{1/2}$$

for E being any one of the three functions concerned.

By this choice of B we get

$$\sum_{q\le\frac{\sqrt{x}}{a\log^{\mathsf{B}}(x)}} \mu^2(q)h^{\omega(q)}\mathcal{E}_E^*(x;aq) \ll \frac{x}{\log^{\mathsf{A}}x},$$

which was to be demonstrated.

Chapter 3

Results due to Selberg's idea

Based on Selberg's idea, we shall invent a multivariable Λ^2 sieve and prove Theorem 1.2.1 in this chapter.

3.1 Preparation

Lemma 3.1.1

Let $z = \frac{N}{\log^{\mathsf{C}} N}$, $\mathsf{C} > 0$, and let $G : \prod_{i=1}^{2\kappa} \mathbb{H}_{-a} \to \mathbb{C}$ be a bounded $\mathcal{C}^\infty$-function. Define

$$J := \frac{1}{(2\pi i)^{2\kappa}} \int_{(1)} \int_{(1)} \cdots \int_{(1)} \int_{(1)} G(s_1, \widetilde{s_1}, \ldots, s_\kappa, \widetilde{s_\kappa}) \times \left[\prod_{i=1}^{\kappa} \frac{\zeta(1+\varphi s_i + \varphi \widetilde{s_i})}{\zeta(1+\varphi s_i)\zeta(1+\varphi\widetilde{s_i})} \frac{z^{s_i}}{s_i^{D_i}} \frac{z^{\widetilde{s_i}}}{\widetilde{s_i}^{\widetilde{D}_i}} \right] ds_1 d\widetilde{s_1} \cdots ds_\kappa d\widetilde{s_\kappa},$$

where $\varphi > 2$ and $D_i, \widetilde{D}_i \geq 2$. Then

$$J = \frac{\varphi^\kappa G(\mathbf{0}) (\log z)^{\sum_1^\kappa (D_i + \widetilde{D}_i - 3)}}{\prod_{i=1}^{\kappa} (D_i - 1)!(\widetilde{D}_i - 1)!} \left[\prod_{i=1}^{\kappa} \frac{(D_i - 1)(\widetilde{D}_i - 1)}{(D_i + \widetilde{D}_i - 3)} \right] + \mathcal{O}\Big((\log N)^{\sum_1^\kappa (D_i + \widetilde{D}_i - 3) - 0.5} \Big),$$

where the implied constant only depends on C, G, φ, D_i and $\widetilde{D}_i$.

Proof (c.f. [7, proof of Lemma 1])

Let $T = e^{\sqrt{\log N}}$. From the c_0 defined in Lemma 2.3.3, we let $c_1 = \frac{c_0}{\varphi}$, $\widetilde{c_1} = \frac{1}{2}\frac{c_0}{\varphi}$, $c_2 = (\frac{1}{2})^2 \frac{c_0}{\varphi}$, $\widetilde{c_2} = (\frac{1}{2})^3 \frac{c_0}{\varphi}$, and so on up to c_κ, $\widetilde{c_\kappa}$. In this way, we have $c_i = \rho_i \frac{c_0}{\varphi}$ and $\widetilde{c_i} = \widetilde{\rho_i} \frac{c_0}{\varphi}$, where $\rho_i = (\frac{1}{2})^{2i-2}$, $\widetilde{\rho_i} = (\frac{1}{2})^{2i-1}$. We also note that Lemma 2.3.3 and Lemma 2.3.4 will be used frequently in the proof.

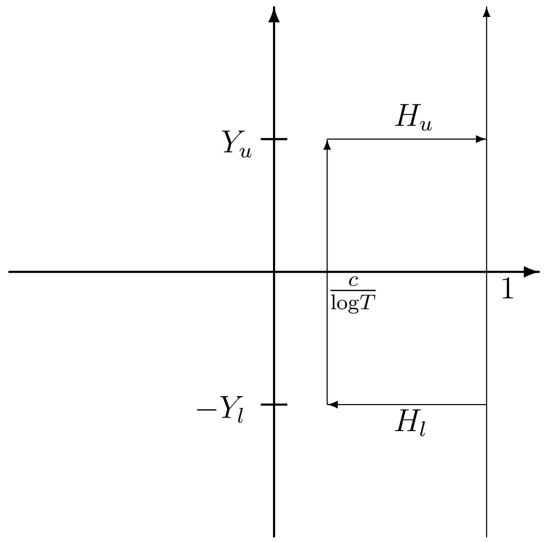

Figure 3.1.1. Note that the rectangle contains no pole of the integrand.

Let's get started. The first step is to shift the s_1-contour of J to $(\frac{c_1}{\log T})$, then the $\widetilde{s_1}$-contour to $(\frac{\widetilde{c_1}}{\log T})$, then the s_2-contour to $(\frac{c_2}{\log T})$, then the $\widetilde{s_2}$-contour to $(\frac{\widetilde{c_2}}{\log T})$, and so on iteratively. This will give

$$J = \frac{1}{(2\pi i)^{2\kappa}} \int\limits_{(\frac{\widetilde{c_\kappa}}{\log T})} \int\limits_{(\frac{c_\kappa}{\log T})} \cdots \int\limits_{(\frac{\widetilde{c_1}}{\log T})} \int\limits_{(\frac{c_1}{\log T})} G(s_1, \widetilde{s_1}, \dots, s_\kappa, \widetilde{s_\kappa}) \times \left[\prod_{i=1}^{\kappa} \frac{\zeta(1+\varphi s_i + \varphi \widetilde{s_i})}{\zeta(1+\varphi s_i)\zeta(1+\varphi \widetilde{s_i})} \frac{z^{s_i}}{s_i^{D_i}} \frac{z^{\widetilde{s_i}}}{\widetilde{s_i}^{\widetilde{D}_i}}\right] ds_1 d\widetilde{s_1} \cdots ds_\kappa d\widetilde{s_\kappa}, \tag{3.1}$$

which can be seen as follows. We have (see Figure 3.1.1)

$$\begin{aligned}\int\limits_{(1)} \cdots \int\limits_{(1)} \int\limits_{(\frac{c}{\log T})} \cdots \int\limits_{(\frac{c}{\log T})} &= \int\limits_{(1)} \cdots \int\limits_{\{\frac{c}{\log T}+it:\, -Y_l \le t \le Y_u\}} \int\limits_{(\frac{c}{\log T})} \cdots \int\limits_{(\frac{c}{\log T})} \\ &+ \int\limits_{(1)} \cdots \int\limits_{H_u \cup H_l} \int\limits_{(\frac{c}{\log T})} \cdots \int\limits_{(\frac{c}{\log T})} \\ &+ \int\limits_{(1)} \cdots \int\limits_{\{1+it:\, t<-Y_l \text{ or } t>Y_u\}} \int\limits_{(\frac{c}{\log T})} \cdots \int\limits_{(\frac{c}{\log T})} .\end{aligned} \tag{3.2}$$

Let $Y = \min\{Y_u, Y_l\}$. The second term at the right of (3.2) is, by $\sum |\frac{\mu(n)}{n^s}| \le \sum \frac{1}{n^\sigma}$,

$$\ll \int\limits_{(1)} \cdots \int\limits_{H_u \cup H_l} \int\limits_{(\frac{c}{\log T})} \cdots \int\limits_{(\frac{c}{\log T})} \left[\prod_{i=1}^{\kappa} \zeta(1+\varphi\sigma_i + \varphi\widetilde{\sigma_i})\zeta(1+\varphi\sigma_i)\zeta(1+\varphi\widetilde{\sigma_i}) \frac{z^{\sigma_i}}{|s_i|^{D_i}} \frac{z^{\widetilde{\sigma_i}}}{|\widetilde{s_i}|^{\widetilde{D}_i}}\right]$$

$$\ll \int\limits_{(1)} \cdots \int\limits_{H_u \cup H_l} \int\limits_{(\frac{c}{\log T})} \cdots \int\limits_{(\frac{c}{\log T})} \left[\prod_{i=1}^{\kappa} (\log^{\mathbb{A}} T) \frac{z^{\sigma_i}}{|s_i|^{D_i}} \frac{z^{\widetilde{\sigma_i}}}{|\widetilde{s_i}|^{\widetilde{D}_i}}\right] \ll \frac{(\log^{\mathbb{A}} T) z^{\mathbb{A}}}{Y},$$

and the third term at the right of (3.2) is

$$\ll \int\limits_{(1)} \cdots \int\limits_{\{1+it:\,|t|>Y\}} \int\limits_{(\frac{c}{\log T})} \cdots \int\limits_{(\frac{c}{\log T})} \left[\prod_{i=1}^{\kappa} (\log^{\mathbb{A}} T) \frac{z^{\sigma_i}}{|s_i|^{D_i}} \frac{z^{\widetilde{\sigma}_i}}{|\widetilde{s}_i|^{\widetilde{D}_i}}\right] \ll \frac{(\log^{\mathbb{A}} T) z^{\mathbb{A}}}{Y}.$$

So by letting $Y_u, Y_l \to \infty$, (3.2) gives

$$\int\limits_{(1)} \cdots \int\limits_{(1)} \int\limits_{(\frac{c}{\log T})} \cdots \int\limits_{(\frac{c}{\log T})} = \int\limits_{(1)} \cdots \int\limits_{(\frac{c}{\log T})} \int\limits_{(\frac{c}{\log T})} \cdots \int\limits_{(\frac{c}{\log T})} .$$

This implies (3.1).

The contours are then truncated to $s_i : |t_i| \le \rho_i T$, $\widetilde{s}_i : |\widetilde{t}_i| \le \widetilde{\rho}_i T$ and denoted by $L_i, \widetilde{L}_i$ respectively. By observing that

$$\begin{aligned} \int\limits_{(\frac{c}{\log T})} \cdots \int\limits_{(\frac{c}{\log T})} \int\limits_{L} \cdots \int\limits_{L} \quad = \quad & \int\limits_{(\frac{c}{\log T})} \cdots \int\limits_{L} \int\limits_{L} \cdots \int\limits_{L} \\ & + \int\limits_{(\frac{c}{\log T})} \cdots \int\limits_{(\frac{c}{\log T})\backslash L} \int\limits_{L} \cdots \int\limits_{L}, \end{aligned}$$

and that the second term at the right is

$$\ll \int\limits_{(\frac{c}{\log T})} \cdots \int\limits_{(\frac{c}{\log T})\backslash L} \int\limits_{L} \cdots \int\limits_{L} \left[\prod_{i=1}^{\kappa} (\log^{\mathbb{A}} T) \frac{z^{\sigma_i}}{|s_i|^{D_i}} \frac{z^{\widetilde{\sigma}_i}}{|\widetilde{s}_i|^{\widetilde{D}_i}}\right] \ll \frac{\log^{\mathbb{A}} T z^{\frac{c}{\log T}}}{T},$$

we see that

$$\begin{aligned} (3.3) \qquad J \quad = \quad & \frac{1}{(2\pi i)^{2\kappa}} \int\limits_{\widetilde{L}_\kappa} \int\limits_{L_\kappa} \cdots \int\limits_{\widetilde{L}_1} \int\limits_{L_1} G(s_1, \widetilde{s}_1, \ldots, s_\kappa, \widetilde{s}_\kappa) \times \\ & \qquad \left[\prod_{i=1}^{\kappa} \frac{\zeta(1+\varphi s_i + \varphi \widetilde{s}_i)}{\zeta(1+\varphi s_i)\zeta(1+\varphi \widetilde{s}_i)} \frac{z^{s_i}}{s_i^{D_i}} \frac{z^{\widetilde{s}_i}}{\widetilde{s}_i^{\widetilde{D}_i}}\right] ds_1 d\widetilde{s}_1 \cdots ds_\kappa d\widetilde{s}_\kappa \\ & + \mathcal{O}\left(\frac{\log^{\mathbb{A}} T z^{\frac{c}{\log T}}}{T}\right). \end{aligned}$$

Next, we go to shift L_1 to $L_1^- : \dfrac{-c_0/\varphi}{\log T} + it_1, |t_1| \le \rho_1 T$. Denoting the integrand at the right of (3.3) by f, we have (see Figure 3.1.2)

(3.4)

$$\begin{aligned} \frac{1}{(2\pi i)^{2\kappa}} \int\limits_{\widetilde{L}_\kappa} \cdots \int\limits_{\widetilde{L}_1} \int\limits_{L_1} f \quad = \quad & \frac{1}{(2\pi i)^{2\kappa}} \int\limits_{\widetilde{L}_\kappa} \cdots \int\limits_{\widetilde{L}_1} \int\limits_{L_1^-} f + \frac{1}{(2\pi i)^{2\kappa}} \int\limits_{\widetilde{L}_\kappa} \cdots \int\limits_{\widetilde{L}_1} \int\limits_{H} f \\ & + \frac{1}{(2\pi i)^{2\kappa-1}} \int\limits_{\widetilde{L}_\kappa} \cdots \int\limits_{\widetilde{L}_1} \left(\left\{\operatorname*{Res}_{s_1=0}\{f\}\right\} + \left\{\operatorname*{Res}_{s_1=-\widetilde{s}_1}\{f\}\right\}\right). \end{aligned}$$

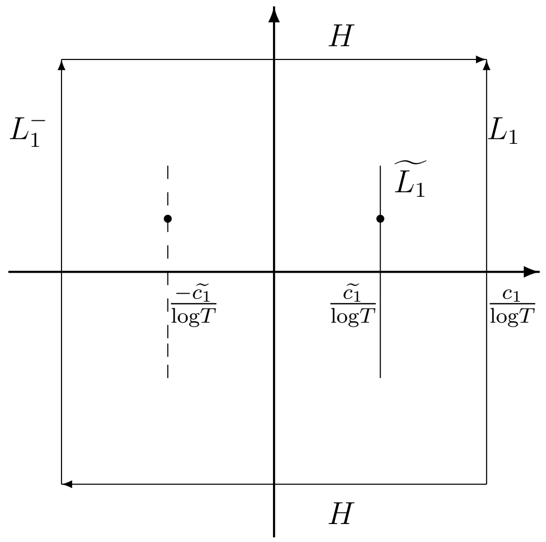

Figure 3.1.2. Each $\widetilde{s_1} \in \widetilde{L_1}$ produces a pole $-\widetilde{s_1}$ which is enclosed by the rectangle.

Note that for each $\widetilde{s_1} \in \widetilde{L_1}$ and each $s_1 \in L_1^-$, $(1+\varphi s_1+\varphi\widetilde{s_1})$ lies on the vertical line $1-\frac{c_0/2}{\log T}+it, |t| \ll T$. Hence, the first term at the right of (3.4) is

$$\ll \int_{\widetilde{L_\kappa}} \cdots \int_{\widetilde{L_1}} \int_{L_1^-} \left[\prod_{i=1}^{\kappa}(\log^{\mathbb{A}} T)\frac{z^{\sigma_i}}{|s_i|^{D_i}}\frac{z^{\widetilde{\sigma_i}}}{|\widetilde{s_i}|^{\widetilde{D}_i}}\right] \ll \log^{\mathbb{A}} T z^{-\frac{c}{\log T}},$$

as $\sum(\sigma_i+\widetilde{\sigma_i}) = \frac{c_0/\varphi}{\log T}(-1+\frac{1}{2}+\frac{1}{4}+\cdots) = -\frac{c}{\log T}$.

By noting that $\mathrm{Im}(1+\varphi s_1+\varphi\widetilde{s_1}) \asymp T$ when $s_1 \in H$, the second term at the right of (3.4) is

$$\ll \int_{\widetilde{L_\kappa}} \cdots \int_{\widetilde{L_1}} \int_{H} \left[\prod_{i=1}^{\kappa}(\log^{\mathbb{A}} T)\frac{z^{\sigma_i}}{|s_i|^{D_i}}\frac{z^{\widetilde{\sigma_i}}}{|\widetilde{s_i}|^{\widetilde{D}_i}}\right] \ll \frac{\log^{\mathbb{A}} T z^{\frac{c}{\log T}}}{T}.$$

So it follows from (3.3) and (3.4) that

$$(3.5)\quad J = \frac{1}{(2\pi i)^{2\kappa-1}}\int_{\widetilde{L_\kappa}} \cdots \int_{\widetilde{L_1}} \left\{\underset{s_1=0}{\mathrm{R}es}\{f\}\right\} + \frac{1}{(2\pi i)^{2\kappa-1}}\int_{\widetilde{L_\kappa}} \cdots \int_{\widetilde{L_1}} \left\{\underset{s_1=-\widetilde{s_1}}{\mathrm{R}es}\{f\}\right\}$$
$$+\mathcal{O}\left(\frac{\log^{\mathbb{A}} T z^{\frac{c}{\log T}}}{T} + \log^{\mathbb{A}} T z^{-\frac{c}{\log T}}\right).$$

We treat the first term at the right by shifting $\widetilde{L_1}$ to $\widetilde{L_1}^- : \dfrac{-c_0/\varphi}{\log T}+i\widetilde{t_1}$, $|\widetilde{t_1}| \le \widetilde{\rho_1}T$. Denote the horizontal contours involved by $\widetilde{H}$. Assuming that

$\widetilde{s_1}=0$ is the only possible pole for $\left\{\underset{s_1=0}{\mathrm{Res}}\{f\}\right\}$, we have

$$\frac{1}{(2\pi i)^{2\kappa-1}}\int\limits_{\widetilde{L_\kappa}}\cdots\int\limits_{L_2}\int\limits_{\widetilde{L_1}}\left\{\underset{s_1=0}{\mathrm{Res}}\{f\}\right\} = \frac{1}{(2\pi i)^{2\kappa-1}}\int\limits_{\widetilde{L_\kappa}}\cdots\int\limits_{L_2}\int\limits_{(\widetilde{L_1}^-\cup\widetilde{H})}\left\{\underset{s_1=0}{\mathrm{Res}}\{f\}\right\} + \frac{1}{(2\pi i)^{2\kappa-2}}\int\limits_{\widetilde{L_\kappa}}\cdots\int\limits_{L_2}\left\{\underset{\widetilde{s_1}=0}{\mathrm{Res}}\left\{\underset{s_1=0}{\mathrm{Res}}\{f\}\right\}\right\}.$$

Let C be the s_1-circle defined by $C:|s_1|=\frac{1}{\log^2 T}$. By noting C does not enclose the point $-\widetilde{s_1}$ for all $\widetilde{s_1}\in\widetilde{L_1}^-\cup\widetilde{H}$, we see that

$$\begin{aligned}\int\limits_{\widetilde{L_\kappa}}\cdots\int\limits_{L_2}\int\limits_{(\widetilde{L_1}^-\cup\widetilde{H})}\left\{\underset{s_1=0}{\mathrm{Res}}\{f\}\right\} &= \frac{1}{2\pi i}\int\limits_{\widetilde{L_\kappa}}\cdots\int\limits_{L_2}\int\limits_{(\widetilde{L_1}^-\cup\widetilde{H})}\int\limits_{C} f\\ &\ll \int\limits_{\widetilde{L_\kappa}}\cdots\int\limits_{L_2}\int\limits_{\widetilde{L_1}^-}\int\limits_{C}\left[\prod_{i=1}^{\kappa}(\log^{\mathbb{A}}T)\frac{z^{\sigma_i}}{|s_i|^{D_i}}\frac{z^{\widetilde{\sigma_i}}}{|\widetilde{s_i}|^{\widetilde{D}_i}}\right]\\ &\quad+\int\limits_{\widetilde{L_\kappa}}\cdots\int\limits_{L_2}\int\limits_{\widetilde{H}}\int\limits_{C}\left[\prod_{i=1}^{\kappa}(\log^{\mathbb{A}}T)\frac{z^{\sigma_i}}{|s_i|^{D_i}}\frac{z^{\widetilde{\sigma_i}}}{|\widetilde{s_i}|^{\widetilde{D}_i}}\right].\end{aligned}$$

Since $z^{\frac{1}{\log^2 T}}\ll 1$, this is $\ll \log^{\mathbb{A}}Tz^{-\frac{c}{\log T}}+\dfrac{\log^{\mathbb{A}}Tz^{\frac{c}{\log T}}}{T}$.

Hence, by defining

$$\begin{aligned}R_{(0)} &:= \left\{\underset{\widetilde{s_1}=0}{\mathrm{Res}}\left\{\underset{s_1=0}{\mathrm{Res}}\{f\}\right\}\right\},\\ R_{(1)} &:= \left\{\underset{s_1=-\widetilde{s_1}}{\mathrm{Res}}\{f\}\right\},\end{aligned}$$

it follows from (3.5) that

$$\begin{aligned}J &= \frac{1}{(2\pi i)^{2\kappa-2}}\int\limits_{\widetilde{L_\kappa}}\cdots\int\limits_{L_2}R_{(0)}+\frac{1}{(2\pi i)^{2\kappa-1}}\int\limits_{\widetilde{L_1}}\int\limits_{\widetilde{L_\kappa}}\cdots\int\limits_{L_2}R_{(1)}\\ &\quad+\mathcal{O}\left(\frac{\log^{\mathbb{A}}Tz^{\frac{c}{\log T}}}{T}+\log^{\mathbb{A}}Tz^{-\frac{c}{\log T}}\right),\end{aligned}\tag{3.6}$$

provided that $\widetilde{s_1}=0$ is the only possible pole for $\left\{\underset{s_1=0}{\mathrm{Res}}\{f\}\right\}$.

By letting

(3.7)

$$Z(s_1,\widetilde{s_1},\ldots,s_\kappa,\widetilde{s_\kappa}) := G(s_1,\widetilde{s_1},\ldots,s_\kappa,\widetilde{s_\kappa})\left[\prod_{i=1}^{\kappa}\frac{(\varphi s_i+\varphi\widetilde{s_i})\zeta(1+\varphi s_i+\varphi\widetilde{s_i})}{\varphi s_i\zeta(1+\varphi s_i)\varphi\widetilde{s_i}\zeta(1+\varphi\widetilde{s_i})}\right],$$

we see that

$$f = Z\left[\prod_{i=1}^{\kappa} \frac{\varphi s_i \widetilde{s_i}}{(s_i+\widetilde{s_i})} \frac{z^{s_i}}{s_i^{D_i}} \frac{z^{\widetilde{s_i}}}{\widetilde{s_i}^{\widetilde{D}_i}}\right] = \left(Z\prod_{i=1}^{\kappa} \varphi z^{s_i} z^{\widetilde{s_i}}\right)\left[\prod_{i=1}^{\kappa} \frac{1}{(s_i+\widetilde{s_i}) s_i^{D_i-1}\widetilde{s_i}^{\widetilde{D}_i-1}}\right],$$

which is of the form stated in Lemma 2.4.3, so (3.6) holds unconditionally. Furthermore, by appealing to that result, we shall shift the contours L_i, $\widetilde{L_i}$ to the left half plane iteratively. Each step in the iteration, which we call Process[▶], is described as follows:

[▶] Let $v \in \{0,1\}^{j-1}$. Follow the definition of $R_v, \theta, \tau, l, \varrho$ in Lemma 2.4.3. When we encounter an integral of the form

$$\frac{1}{(2\pi i)^{2\kappa-2\varrho-l}} \int\limits_{\widetilde{L_{\tau_l}}} \cdots \int\limits_{\widetilde{L_{\tau_1}}} \int\limits_{\widetilde{L_\kappa}} \int\limits_{L_\kappa} \cdots \int\limits_{\widetilde{L_j}} \int\limits_{L_j} R_v, \tag{3.8}$$

we first shift L_j to $L_j^- : \dfrac{-c_0/\varphi}{\log T} + it_j, |t_j| \le \rho_j T$. Denoting the horizontal contours involved by H, this results in

$$\int\limits_{\widetilde{L_{\tau_l}}} \cdots \int\limits_{\widetilde{L_{\tau_1}}} \int\limits_{\widetilde{L_\kappa}} \cdots \int\limits_{\widetilde{L_j}} \int\limits_{(L_j^- \cup H)} \frac{R_v}{(2\pi i)^{2\kappa-2\varrho-l}} + \int\limits_{\widetilde{L_{\tau_l}}} \cdots \int\limits_{\widetilde{L_{\tau_1}}} \int\limits_{\widetilde{L_\kappa}} \cdots \int\limits_{\widetilde{L_j}} \frac{\left(\operatorname*{Res}_{s_j=0} + \operatorname*{Res}_{s_j=-\widetilde{s_j}}\right)\{R_v\}}{(2\pi i)^{2\kappa-2\varrho-l-1}}.$$

We go to bound the first term. Let $\widetilde{C_\theta}$ be the $\widetilde{s_\theta}$-circle $|\widetilde{s_\theta}| = \frac{1/\widetilde{\rho_\theta}}{\log^2 T}$, C_θ be the s_θ-circle $|s_\theta| = \frac{1/\rho_\theta}{\log^2 T}$, where $\theta = \theta_1, \ldots, \theta_\varrho$; and for each $(\widetilde{s_{\tau_1}}, \ldots, \widetilde{s_{\tau_l}})$ in $\widetilde{L_{\tau_1}} \times \cdots \times \widetilde{L_{\tau_l}}$, let C_τ be the s_τ-circle $C_\tau : |s_\tau + \widetilde{s_\tau}| = \frac{1/\rho_\tau}{\log^2 T}$, where $\tau = \tau_1, \ldots, \tau_l$.

The term under consideration is then

$$= \frac{1}{(2\pi i)^{2\kappa}} \int\limits_{\widetilde{L_{\tau_l}}} \cdots \int\limits_{\widetilde{L_{\tau_1}}} \int\limits_{\widetilde{L_\kappa}} \cdots \int\limits_{\widetilde{L_j}} \int\limits_{(L_j^- \cup H)} \int\limits_{C_{\tau_l}} \cdots \int\limits_{C_{\tau_1}} \int\limits_{\widetilde{C_{\theta_\varrho}}} \int\limits_{C_{\theta_\varrho}} \cdots \int\limits_{\widetilde{C_{\theta_1}}} \int\limits_{C_{\theta_1}} f,$$

which is $\ll \log^{\mathbb{A}} T z^{-\frac{c}{\log T}} + \dfrac{\log^{\mathbb{A}} T z^{\frac{c}{\log T}}}{T}$.

Therefore, the integral (3.8) is

$$= \int\limits_{\widetilde{L_{\tau_l}}} \cdots \int\limits_{\widetilde{L_{\tau_1}}} \int\limits_{\widetilde{L_\kappa}} \cdots \int\limits_{\widetilde{L_j}} \frac{\left\{\operatorname*{Res}_{s_j=0}\{R_v\}\right\}}{(2\pi i)^{2\kappa-2\varrho-l-1}} + \int\limits_{\widetilde{L_j}} \int\limits_{\widetilde{L_{\tau_l}}} \cdots \int\limits_{\widetilde{L_{\tau_1}}} \int\limits_{\widetilde{L_\kappa}} \cdots \int\limits_{L_{j+1}} \frac{R_{(v,1)}}{(2\pi i)^{2\kappa-2\varrho-(l+1)}}$$

$$+ \quad \mathcal{O}\left(\frac{\log^{\mathbb{A}} T z^{\frac{c}{\log T}}}{T} + \log^{\mathbb{A}} T z^{-\frac{c}{\log T}}\right).$$

We then treat the first term in the last expression by shifting $\widetilde{L_j}$ to $\widetilde{L_j}^- : \dfrac{-c_0/\varphi}{\log T} + i\widetilde{t_j},\ |\widetilde{t_j}| \leq \widetilde{\rho_j}T$. Denoting the horizontal contours involved by $\widetilde{H}$,

$$\int\limits_{\widetilde{L_{\tau_l}}} \cdots \int\limits_{\widetilde{L_{\tau_1}}} \int\limits_{\widetilde{L_\kappa}} \cdots \int\limits_{\widetilde{L_j}} \frac{\left\{\underset{s_j=0}{\mathrm{Res}}\{R_v\}\right\}}{(2\pi i)^{2\kappa-2\varrho-l-1}} = \int\limits_{\widetilde{L_{\tau_l}}} \cdots \int\limits_{\widetilde{L_{\tau_1}}} \int\limits_{\widetilde{L_\kappa}} \cdots \int\limits_{(\widetilde{L_j}^- \cup \widetilde{H})} \frac{\left\{\underset{s_j=0}{\mathrm{Res}}\{R_v\}\right\}}{(2\pi i)^{2\kappa-2\varrho-l-1}} + \int\limits_{\widetilde{L_{\tau_l}}} \cdots \int\limits_{\widetilde{L_{\tau_1}}} \int\limits_{\widetilde{L_\kappa}} \cdots \int\limits_{L_{j+1}} \frac{R_{(v,0)}}{(2\pi i)^{2\kappa-2(\varrho+1)-l}}.$$

Let C_j be the s_j-circle defined by $C_j : |s_j| = \frac{1/p_j}{\log^2 T}$. By noting C_j does not enclose the point $-\widetilde{s_j}$ for all $\widetilde{s_j} \in \widetilde{L_j}^- \cup \widetilde{H}$, we see that

$$\begin{aligned}
&\int\limits_{\widetilde{L_{\tau_l}}} \cdots \int\limits_{\widetilde{L_{\tau_1}}} \int\limits_{\widetilde{L_\kappa}} \cdots \int\limits_{(\widetilde{L_j}^- \cup \widetilde{H})} \frac{\left\{\underset{s_j=0}{\mathrm{Res}}\right\}}{(2\pi i)^{2\kappa-2\varrho-l-1}} \\
&= \frac{1}{(2\pi i)^{2\kappa}} \int\limits_{\widetilde{L_{\tau_l}}} \cdots \int\limits_{\widetilde{L_{\tau_1}}} \int\limits_{\widetilde{L_\kappa}} \cdots \int\limits_{(\widetilde{L_j}^- \cup \widetilde{H})} \int\limits_{C_j} \int\limits_{C_{\tau_l}} \cdots \int\limits_{C_{\tau_1}} \int\limits_{\widetilde{C_{\theta_\varrho}}} \int\limits_{C_{\theta_\varrho}} \cdots \int\limits_{\widetilde{C_{\theta_1}}} \int\limits_{C_{\theta_1}} f \\
&\ll \log^{\mathbb{A}} T z^{-\frac{c}{\log T}} + \frac{\log^{\mathbb{A}} T z^{\frac{c}{\log T}}}{T}.
\end{aligned}$$

Hence, in summary, the integral (3.8) is

$$\begin{aligned}
&= \int\limits_{\widetilde{L_{\tau_l}}} \cdots \int\limits_{\widetilde{L_{\tau_1}}} \int\limits_{\widetilde{L_\kappa}} \cdots \int\limits_{L_{j+1}} \frac{R_{(v,0)}}{(2\pi i)^{2\kappa-2(\varrho+1)-l}} + \int\limits_{\widetilde{L_j}} \int\limits_{\widetilde{L_{\tau_l}}} \cdots \int\limits_{\widetilde{L_{\tau_1}}} \int\limits_{\widetilde{L_\kappa}} \cdots \int\limits_{L_{j+1}} \frac{R_{(v,1)}}{(2\pi i)^{2\kappa-2\varrho-(l+1)}} \\
&\quad + \mathcal{O}\left(\frac{\log^{\mathbb{A}} T z^{\frac{c}{\log T}}}{T} + \log^{\mathbb{A}} T z^{-\frac{c}{\log T}} \right).
\end{aligned}$$

This ends the process.

After iteration of Process[▶], (3.6) gives

$$J = \sum_{v \in \{0,1\}^\kappa} \frac{1}{(2\pi i)^l} \int\limits_{\widetilde{L_{\tau_l}}} \cdots \int\limits_{\widetilde{L_{\tau_1}}} R_v + \mathcal{O}\left(\frac{\log^{\mathbb{A}} T z^{\frac{c}{\log T}}}{T} + \log^{\mathbb{A}} T z^{-\frac{c}{\log T}} \right).$$

For each $v \in \{0,1\}^\kappa$, $v \neq (0,\ldots,0)$, we have

$$
\begin{aligned}
& \frac{1}{(2\pi i)^l} \int\limits_{\widetilde{L_{\tau_l}}} \cdots \int\limits_{\widetilde{L_{\tau_1}}} R_v \\
= \quad & \frac{1}{(2\pi i)^{2\kappa}} \int\limits_{\widetilde{L_{\tau_l}}} \cdots \int\limits_{\widetilde{L_{\tau_1}}} \int\limits_{C_{\tau_l}} \cdots \int\limits_{C_{\tau_1}} \int\limits_{\widetilde{C_{\theta_\varrho}}} \int\limits_{C_{\theta_\varrho}} \cdots \int\limits_{\widetilde{C_{\theta_1}}} \int\limits_{C_{\theta_1}} \\
& \qquad G\left[\prod_{i=1}^{\kappa} \frac{\zeta(1+\varphi s_i+\varphi\widetilde{s_i})}{\zeta(1+\varphi s_i)\zeta(1+\varphi\widetilde{s_i})} \frac{z^{s_i}}{s_i^{D_i}} \frac{z^{\widetilde{s_i}}}{\widetilde{s_i}^{\widetilde{D_i}}}\right] ds_1 d\widetilde{s_1} \cdots ds_\kappa d\widetilde{s_\kappa} \\
\ll \quad & \int\limits_{\widetilde{L_{\tau_l}}} \cdots \int\limits_{\widetilde{L_{\tau_1}}} \int\limits_{C_{\tau_l}} \cdots \int\limits_{C_{\tau_1}} \int\limits_{\widetilde{C_{\theta_\varrho}}} \int\limits_{C_{\theta_\varrho}} \cdots \int\limits_{\widetilde{C_{\theta_1}}} \int\limits_{C_{\theta_1}} \\
& \left[\prod_{\theta=\theta_1,\ldots,\theta_\varrho} \log^2 T \frac{1}{|s_\theta|^{D_\theta - 1}} \frac{1}{|\widetilde{s_\theta}|^{\widetilde{D}_\theta - 1}}\right]\left[\prod_{\tau=\tau_1,\ldots,\tau_l} \log^2 T \frac{1}{|\widetilde{s_\tau}|^{D_\tau+\widetilde{D}_\tau - 2}}\right] |ds_1| \cdots |d\widetilde{s_\kappa}| \\
\ll \quad & (\log T)^{-2\varrho + \sum_\theta (2D_\theta + 2\widetilde{D}_\theta - 4)} (\log T)^{\sum_\tau (D_\tau + \widetilde{D}_\tau - 3)} \\
= \quad & (\log T)^{\sum_\theta (2D_\theta + 2\widetilde{D}_\theta) + \sum_\tau (D_\tau + \widetilde{D}_\tau) - 6\kappa + 3l} \qquad \because \varrho + l = \kappa \\
= \quad & (\log N)^{\sum_1^\kappa (D_i + \widetilde{D}_i - 3) - \sum_\tau (\frac{D_\tau + \widetilde{D}_\tau - 3}{2})} \qquad \because \log T = (\log N)^{\frac{1}{2}} \\
\ll \quad & (\log N)^{\sum_1^\kappa (D_i + \widetilde{D}_i - 3) - 0.5}.
\end{aligned}
$$

It follows that

$$
J \;=\; R_{(0,\ldots,0)} + \mathcal{O}\Big((\log N)^{\sum_1^\kappa (D_i + \widetilde{D}_i - 3) - 0.5}\Big). \tag{3.9}
$$

Let $\widetilde{C_i}$ be the $\widetilde{s_i}$-circle $|\widetilde{s_i}| = (1/\widetilde{\rho_i})r$, C_i be the s_i-circle $|s_i| = (1/\rho_i)r$, where r is a suitably small number. Now

$$
R_{(0,\ldots,0)} = \frac{1}{(2\pi i)^{2\kappa}} \int\limits_{\widetilde{C_\kappa}} \int\limits_{C_\kappa} \cdots \int\limits_{\widetilde{C_1}} \int\limits_{C_1} G\left[\prod_{i=1}^{\kappa} \frac{\zeta(1+\varphi s_i+\varphi\widetilde{s_i})}{\zeta(1+\varphi s_i)\zeta(1+\varphi\widetilde{s_i})} \frac{z^{s_i}}{s_i^{D_i}} \frac{z^{\widetilde{s_i}}}{\widetilde{s_i}^{\widetilde{D_i}}}\right] ds_1 \cdots d\widetilde{s_\kappa},
$$

in which we have used the fact that for each $\widetilde{s_i} \in \widetilde{C_i}$, if $|s_i| \leq (1/\rho_i)r$, then $|s_i + \widetilde{s_i}| \geq |\widetilde{s_i}| - |s_i| \geq (1/\widetilde{\rho_i})r - (1/\rho_i)r > 0$, so C_i does not enclose the pole $s_i = -\widetilde{s_i}$. Note that the choice of r only depends on G and φ but is independent of z.

Recall the definition of $Z(s_1,\widetilde{s_1},\ldots,s_\kappa,\widetilde{s_\kappa})$ in (3.7). We have

$$R_{(0,\ldots,0)} = \frac{1}{(2\pi i)^{2\kappa}}\int\limits_{\widetilde{C_\kappa}}\int\limits_{C_\kappa}\cdots\int\limits_{\widetilde{C_1}}\int\limits_{C_1} Z\left[\prod_{i=1}^{\kappa}\frac{\varphi s_i\widetilde{s_i}}{(s_i+\widetilde{s_i})}\frac{z^{s_i}}{s_i^{D_i}}\frac{z^{\widetilde{s_i}}}{\widetilde{s_i}^{\widetilde{D}_i}}\right]ds_1\cdots d\widetilde{s_\kappa}.$$

By Lemma 2.4.5, this integral equals

$$\frac{1}{(2\pi i)^{2\kappa}}\int\limits_{\widetilde{C_\kappa}}\int\limits_{C'_\kappa}\cdots\int\limits_{\widetilde{C'_1}}\int\limits_{C'_1}\frac{Z(\widetilde{s_\kappa}\xi_1,\widetilde{s_\kappa}\widetilde{\xi_1},\ldots,\widetilde{s_\kappa}\xi_\kappa,\widetilde{s_\kappa})}{\widetilde{s_\kappa}^{\sum_1^\kappa(D_i+\widetilde{D}_i)-\kappa-(2\kappa-1)}}\times$$
$$\left[\prod_{i=1}^{\kappa-1}\frac{\varphi\xi_i\widetilde{\xi_i}}{(\xi_i+\widetilde{\xi_i})}\frac{z^{\widetilde{s_\kappa}(\xi_i+\widetilde{\xi_i})}}{\xi_i^{D_i}\widetilde{\xi_i}^{\widetilde{D}_i}}\right]\left(\frac{\varphi\xi_\kappa}{(\xi_\kappa+1)}\frac{z^{\widetilde{s_\kappa}(\xi_\kappa+1)}}{\xi_\kappa^{D_\kappa}}\right)d\xi_1 d\widetilde{\xi_1}\cdots d\xi_\kappa d\widetilde{s_\kappa},$$

where $\widetilde{C'_i}$ is the circle $|\widetilde{\xi_i}| = \widetilde{\rho_\kappa}/\widetilde{\rho_i}$ for $1\le i\le\kappa-1$, C'_i is the circle $|\xi_i| = \widetilde{\rho_\kappa}/\rho_i$ for $1\le i\le\kappa$.

For each $(\xi_1,\widetilde{\xi_1},\ldots,\xi_\kappa)\in C'_1\times\widetilde{C'_1}\times\cdots\times C'_\kappa$, $Z(\widetilde{s_\kappa}\xi_1,\widetilde{s_\kappa}\widetilde{\xi_1},\ldots,\widetilde{s_\kappa}\xi_\kappa,\widetilde{s_\kappa})$ is an analytic function in $\widetilde{s_\kappa}$ on $|\widetilde{s_\kappa}|<(1/\widetilde{\rho_\kappa})r$, and uniformly in $(\xi_1,\ldots,\xi_\kappa)$ is $\ll 1$ in this domain. Hence, for any fixed $n\in\mathbb{N}$, by Cauchy's estimation, uniformly in $(\xi_1,\ldots,\xi_\kappa)$ we have $\left.\dfrac{\partial^n Z}{\partial\widetilde{s_\kappa}^n}\right|_{\widetilde{s_\kappa}=0}\ll r^{-n}\ll 1$ as well. Therefore, by changing the order of integration, the last integral is

$$\begin{aligned}
=\;& \frac{\varphi^\kappa}{(2\pi i)^{2\kappa}}\int\limits_{C'_\kappa}\cdots\int\limits_{C'_1}\int\limits_{\widetilde{C_\kappa}}\left[\prod_{i=1}^{\kappa-1}\frac{\xi_i\widetilde{\xi_i}}{(\xi_i+\widetilde{\xi_i})}\frac{1}{\xi_i^{D_i}\widetilde{\xi_i}^{\widetilde{D}_i}}\right]\left(\frac{\xi_\kappa}{(\xi_\kappa+1)}\frac{1}{\xi_\kappa^{D_\kappa}}\right)\times\\
&\qquad\frac{Z(\widetilde{s_\kappa}\xi_1,\widetilde{s_\kappa}\widetilde{\xi_1},\ldots,\widetilde{s_\kappa}\xi_\kappa,\widetilde{s_\kappa})}{\widetilde{s_\kappa}^{\sum_1^\kappa(D_i+\widetilde{D}_i)-3\kappa+1}}z^{\widetilde{s_\kappa}[\sum_1^{\kappa-1}(\xi_i+\widetilde{\xi_i})+\xi_\kappa+1]}\,d\widetilde{s_\kappa}d\xi_1 d\widetilde{\xi_1}\cdots d\xi_\kappa\\
=\;& \frac{\varphi^\kappa Z(\mathbf{0})(\log z)^{\sum_1^\kappa(D_i+\widetilde{D}_i-3)}}{(\sum_1^\kappa(D_i+\widetilde{D}_i-3))!(2\pi i)^{2\kappa-1}}\int\limits_{C'_\kappa}\cdots\int\limits_{C'_1}\left[\sum_1^{\kappa-1}(\xi_i+\widetilde{\xi_i})+\xi_\kappa+1\right]^{\sum_1^\kappa(D_i+\widetilde{D}_i-3)}\\
&\qquad\times\left[\prod_{i=1}^{\kappa-1}\frac{1}{(\xi_i+\widetilde{\xi_i})\xi_i^{D_i-1}\widetilde{\xi_i}^{\widetilde{D}_i-1}}\right]\left(\frac{1}{(\xi_\kappa+1)\xi_\kappa^{D_\kappa-1}}\right)d\xi_1 d\widetilde{\xi_1}\cdots d\xi_\kappa\\
&+\mathcal{O}\left(\int\limits_{C'_\kappa}\cdots\int\limits_{C'_1}\frac{\left(\left[\sum_1^{\kappa-1}(|\xi_i|+|\widetilde{\xi_i}|)+|\xi_\kappa|+1\right]\log z\right)^{\sum_1^\kappa(D_i+\widetilde{D}_i-3)-1}}{\left[\prod_{i=1}^{\kappa-1}(|\widetilde{\xi_i}|-|\xi_i|)|\xi_i|^{D_i-1}|\widetilde{\xi_i}|^{\widetilde{D}_i-1}\right](1-|\xi_\kappa|)|\xi_\kappa|^{D_\kappa-1}}\right).
\end{aligned}$$

The $\mathcal{O}$-term is $\ll(\log N)^{\sum_1^\kappa(D_i+\widetilde{D}_i-3)-1}$.

The main term is, by $Z(\mathbf{0}) = G(\mathbf{0})$ and multinomial expansion,

$$\begin{aligned}
&= \frac{\varphi^\kappa G(\mathbf{0})(\log z)^{\sum_1^\kappa (D_i+\widetilde{D}_i-3)}}{(\sum_1^\kappa (D_i+\widetilde{D}_i-3))!} \frac{1}{(2\pi i)^{2\kappa-1}} \int_{C'_\kappa} \cdots \int_{\widetilde{C}'_1} \int_{C'_1} \\
&\qquad \times \sum_{j_1,\ldots,j_\kappa} \frac{(\sum_1^\kappa (D_i+\widetilde{D}_i-3))!}{j_1!\cdots j_\kappa!} (\xi_1+\widetilde{\xi}_1)^{j_1}\cdots(\xi_\kappa+1)^{j_\kappa} \\
&\qquad \times \left[\prod_{i=1}^{\kappa-1} \frac{1}{(\xi_i+\widetilde{\xi}_i)\xi_i^{D_i-1}\widetilde{\xi}_i^{\widetilde{D}_i-1}}\right] \left(\frac{1}{(\xi_\kappa+1)\xi_\kappa^{D_\kappa-1}}\right) d\xi_1 d\widetilde{\xi}_1\cdots d\xi_\kappa \\
&= \sum_{j_1,\ldots,j_\kappa} \frac{\varphi^\kappa G(\mathbf{0})(\log z)^{\sum_1^\kappa (D_i+\widetilde{D}_i-3)}}{j_1!\cdots j_\kappa!} \times \\
&\qquad \frac{1}{(2\pi i)^{2\kappa-1}} \int_{C'_\kappa} \cdots \int_{\widetilde{C}'_1} \int_{C'_1} \left[\prod_{i=1}^{\kappa-1} \frac{(\xi_i+\widetilde{\xi}_i)^{j_i-1}}{\xi_i^{D_i-1}\widetilde{\xi}_i^{\widetilde{D}_i-1}}\right] \frac{(\xi_\kappa+1)^{j_\kappa-1}}{\xi_\kappa^{D_\kappa-1}} d\xi_1 d\widetilde{\xi}_1\cdots d\xi_\kappa.
\end{aligned}$$

For $1 \le i \le \kappa-1$, if $j_i - 1 \ne D_i - 2 + \widetilde{D}_i - 2$, then the expansion of $\dfrac{(\xi_i+\widetilde{\xi}_i)^{j_i-1}}{\xi_i^{D_i-1}\widetilde{\xi}_i^{\widetilde{D}_i-1}}$ does not contain the $\dfrac{1}{\xi_i\widetilde{\xi}_i}$-term; else if $j_i = D_i + \widetilde{D}_i - 3$, we have the $\dfrac{1}{\xi_i\widetilde{\xi}_i}$-term with coefficient $\mathrm{C}_{D_i-2}^{D_i+\widetilde{D}_i-4}$. It follows that the expression above is

$$= \frac{\varphi^\kappa G(\mathbf{0})(\log z)^{\sum_1^\kappa (D_i+\widetilde{D}_i-3)}}{(D_1+\widetilde{D}_1-3)!\cdots j_\kappa!} \frac{1}{2\pi i}\int_{C'_\kappa} \left[\prod_{i=1}^{\kappa-1} \mathrm{C}_{D_i-2}^{D_i+\widetilde{D}_i-4}\right] \frac{(\xi_\kappa+1)^{j_\kappa-1}}{\xi_\kappa^{D_\kappa-1}} d\xi_\kappa,$$

where $j_\kappa = \sum_1^\kappa (D_i+\widetilde{D}_i-3) - \sum_1^{\kappa-1}(D_i+\widetilde{D}_i-3) = D_\kappa + \widetilde{D}_\kappa - 3$, so this is

$$\begin{aligned}
&= \frac{\varphi^\kappa G(\mathbf{0})(\log z)^{\sum_1^\kappa (D_i+\widetilde{D}_i-3)}}{\prod_{i=1}^\kappa (D_i+\widetilde{D}_i-3)!} \frac{1}{2\pi i}\int_{C'_\kappa} \left[\prod_{i=1}^{\kappa-1} \mathrm{C}_{D_i-2}^{D_i+\widetilde{D}_i-4}\right] \frac{(\xi_\kappa+1)^{D_\kappa+\widetilde{D}_\kappa-4}}{\xi_\kappa^{D_\kappa-1}} d\xi_\kappa \\
&= \frac{\varphi^\kappa G(\mathbf{0})(\log z)^{\sum_1^\kappa (D_i+\widetilde{D}_i-3)}}{\prod_{i=1}^\kappa (D_i+\widetilde{D}_i-3)!} \left[\prod_{i=1}^{\kappa} \mathrm{C}_{D_i-2}^{D_i+\widetilde{D}_i-4}\right] \\
&= \frac{\varphi^\kappa G(\mathbf{0})(\log z)^{\sum_1^\kappa (D_i+\widetilde{D}_i-3)}}{\prod_{i=1}^\kappa (D_i-1)!(\widetilde{D}_i-1)!} \left[\prod_{i=1}^{\kappa} \frac{(D_i-1)(\widetilde{D}_i-1)}{(D_i+\widetilde{D}_i-3)}\right].
\end{aligned}$$

We conclude from (3.9) that

$$\begin{aligned}
J \;&=\; \frac{\varphi^\kappa G(\mathbf{0})(\log z)^{\sum_1^\kappa (D_i+\widetilde{D}_i-3)}}{\prod_{i=1}^\kappa (D_i-1)!(\widetilde{D}_i-1)!} \left[\prod_{i=1}^{\kappa} \frac{(D_i-1)(\widetilde{D}_i-1)}{(D_i+\widetilde{D}_i-3)}\right] \\
&\quad + \mathcal{O}\Big((\log N)^{\sum_1^\kappa (D_i+\widetilde{D}_i-3)-0.5}\Big).
\end{aligned}$$

Q.E.D.

3.2 The choice of λ and the asymptotic formula for Q_0

Let $D_1, D_2 \in \mathbb{N}$ and $\varphi > 2$. We shall choose the following λ for the case $k = 2$:

Definition 3.2.1

$$\lambda_{\delta_1,\delta_2}^{D_1,D_2} := \begin{cases} \mu(\delta_1)\mu(\delta_2)\left(\dfrac{\log\frac{z}{\delta_1^{\varphi}}}{\log z}\right)^{D_1-1}\left(\dfrac{\log\frac{z}{\delta_2^{\varphi}}}{\log z}\right)^{D_2-1} & \textit{if } \delta_1^{\varphi}, \delta_2^{\varphi} \le z := \frac{N}{\log^{\mathsf{C}} N} \\ 0 & \textit{otherwise,} \end{cases}$$

where $\mathsf{C} = 2\mathsf{B}(3,3) + 3$, *and* $\mathsf{B}(3,3)$ *is the one given by Proposition 2.5.1.*

By varying N if necessary, throughout this chapter we assume that $z^{1/\varphi} \notin \mathbb{N}$. Also, recall Definition 2.1.2 and the discussion of $\mathfrak{S}$ right after it.

Theorem 3.2.2

Let $\{a_1n+b_1, a_2n+b_2\}$ *be HB-admissible, and let*

$$Q_0 := \sum_{N<n\le 2N}\left(\sum_{\substack{\delta_1|a_1n+b_1\\ \delta_2|a_2n+b_2}} \lambda_{\delta_1,\delta_2}^{D_1,D_2}\right)\left(\sum_{\substack{\widetilde{\delta}_1|a_1n+b_1\\ \widetilde{\delta}_2|a_2n+b_2}} \lambda_{\widetilde{\delta}_1,\widetilde{\delta}_2}^{\widetilde{D}_1,\widetilde{D}_2}\right),$$

where $\lambda_{\delta_1,\delta_2}^{\mathbf{D}}$ *is given by Definition 3.2.1. Suppose* D_1, D_2, $\widetilde{D}_1$, $\widetilde{D}_2 \ge 2$. *Then for any* $\varphi \ge 4$, *we have*

$$Q_0 = \frac{\varphi^2\mathfrak{S}N}{(\log N)^2}\left[\prod_{i=1}^{2}\frac{(D_i-1)(\widetilde{D}_i-1)}{(D_i+\widetilde{D}_i-3)}\right] + \mathcal{O}\left(\frac{N}{(\log N)^{2.5}}\right),$$

where the implied constant only depends on $\mathfrak{L}$, D_1, D_2, $\widetilde{D}_1$, $\widetilde{D}_2$ *and* φ.

Proof

By Theorem 2.2.2(a) with $k = 2$, we have

$$Q_0 = \sum_{\substack{\delta_1,\delta_2\\ \widetilde{\delta}_1,\widetilde{\delta}_2\\ (\Delta_1,\Delta_2)=1\\ (\Delta_1\Delta_2,A)=1}} \lambda\widetilde{\lambda}\frac{N}{\Delta_1\Delta_2} + \mathcal{O}\left(\sum_{\substack{\delta_1,\delta_2\\ \widetilde{\delta}_1,\widetilde{\delta}_2}} 1\right).$$

The $\mathcal{O}$-term is $\ll (z^{1/\varphi})^4 \ll z$, which is acceptable.

The main term is

$$= \frac{N}{(\log z)^{D_1+\widetilde{D}_1+D_2+\widetilde{D}_2-4}} \times \sum_{\substack{\delta_1,\delta_2\\ \widetilde{\delta}_1,\widetilde{\delta}_2\\ (\Delta_1,\Delta_2)=1\\ (\Delta_1\Delta_2,A)=1}} \frac{[\mu(\delta_1\delta_2)][\mu(\widetilde{\delta}_1\widetilde{\delta}_2)]}{\Delta_1\Delta_2}\left(\log\frac{z}{\delta_1^{\varphi}}\right)^{D_1-1}\left(\log\frac{z}{\widetilde{\delta}_1^{\varphi}}\right)^{\widetilde{D}_1-1}\left(\log\frac{z}{\delta_2^{\varphi}}\right)^{D_2-1}\left(\log\frac{z}{\widetilde{\delta}_2^{\varphi}}\right)^{\widetilde{D}_2-1},$$

which is

(3.10)

$$= \frac{N}{(\log z)^{D_1+\widetilde{D}_1+D_2+\widetilde{D}_2-4}} \times \frac{(D_1-1)!(\widetilde{D}_1-1)!(D_2-1)!(\widetilde{D}_2-1)!}{(2\pi i)^4}\int_{(1)}\int_{(1)}\int_{(1)}\int_{(1)} F\frac{z^{s_1}}{s_1^{D_1}}\frac{z^{\widetilde{s}_1}}{\widetilde{s}_1^{\widetilde{D}_1}}\frac{z^{s_2}}{s_2^{D_2}}\frac{z^{\widetilde{s}_2}}{\widetilde{s}_2^{\widetilde{D}_2}}ds_1d\widetilde{s}_1ds_2d\widetilde{s}_2,$$

where

$$\begin{aligned} F &:= \sum_{\substack{\delta_1,\delta_2\\ \widetilde{\delta}_1,\widetilde{\delta}_2\\ (\Delta_1,\Delta_2)=1\\ (\Delta_1\Delta_2,A)=1}}^{\infty} \frac{[\mu(\delta_1\delta_2)][\mu(\widetilde{\delta}_1\widetilde{\delta}_2)]}{\Delta_1\Delta_2}\frac{1}{(\delta_1)^{\varphi s_1}}\frac{1}{(\widetilde{\delta}_1)^{\varphi\widetilde{s}_1}}\frac{1}{(\delta_2)^{\varphi s_2}}\frac{1}{(\widetilde{\delta}_2)^{\varphi\widetilde{s}_2}} \\ &= \prod_{p\nmid A}\left[1-\frac{1}{p}\left(\frac{1}{p^{\varphi s_1}}+\frac{1}{p^{\varphi\widetilde{s}_1}}-\frac{1}{p^{\varphi s_1+\varphi\widetilde{s}_1}}+\frac{1}{p^{\varphi s_2}}+\frac{1}{p^{\varphi\widetilde{s}_2}}-\frac{1}{p^{\varphi s_2+\varphi\widetilde{s}_2}}\right)\right]. \end{aligned}$$

Note that we have used Lemma 2.4.4 in (3.10), as $z^{1/\varphi}\notin\mathbb{N}$ and

$$\sum_{\substack{\delta_1,\delta_2\\ \widetilde{\delta}_1,\widetilde{\delta}_2\\ (\Delta_1,\Delta_2)=1\\ (\Delta_1\Delta_2,A)=1}}^{\infty} \frac{|\mu(\delta_1\delta_2)\mu(\widetilde{\delta}_1\widetilde{\delta}_2)|}{\Delta_1\Delta_2\cdot\delta_1^{\varphi}\delta_2^{\varphi}\widetilde{\delta}_1^{\varphi}\widetilde{\delta}_2^{\varphi}} = \prod_{p\nmid A}\left[1+\frac{1}{p}\left(\frac{2}{p^{\varphi}}+\frac{2}{p^{\varphi}}+\frac{2}{p^{2\varphi}}\right)\right]<\infty.$$

Let

$$\begin{aligned} g(p) &:= \left(1-\frac{1}{p^{1+\varphi s_1}}\right)^{-1}\left(1-\frac{1}{p^{1+\varphi\widetilde{s}_1}}\right)^{-1}\left(1-\frac{1}{p^{1+\varphi s_1+\varphi\widetilde{s}_1}}\right)\times \\ &\quad \left(1-\frac{1}{p^{1+\varphi s_2}}\right)^{-1}\left(1-\frac{1}{p^{1+\varphi\widetilde{s}_2}}\right)^{-1}\left(1-\frac{1}{p^{1+\varphi s_2+\varphi\widetilde{s}_2}}\right), \end{aligned}$$

so that

$$\begin{aligned} F &= \left\{F\cdot\prod_p g(p)\right\}\prod_p g(p)^{-1} \\ &:= \{G\}\frac{\zeta(1+\varphi s_1+\varphi\widetilde{s}_1)\zeta(1+\varphi s_2+\varphi\widetilde{s}_2)}{\zeta(1+\varphi s_1)\zeta(1+\varphi\widetilde{s}_1)\zeta(1+\varphi s_2)\zeta(1+\varphi\widetilde{s}_2)}. \end{aligned}$$

By Lemma 2.3.5, Proposition 2.4.2, and Lemma 3.1.1, (3.10) is

$$= \frac{N(D_1-1)!(\widetilde{D}_1-1)!(D_2-1)!(\widetilde{D}_2-1)!}{(\log z)^{D_1+\widetilde{D}_1+D_2+\widetilde{D}_2-4}} \times$$

$$\frac{1}{(2\pi i)^4}\int_{(1)}\int_{(1)}\int_{(1)}\int_{(1)} G\left[\prod_{i=1}^{2}\frac{\zeta(1+\varphi s_i+\varphi\widetilde{s}_i)}{\zeta(1+\varphi s_i)\zeta(1+\varphi\widetilde{s}_i)}\frac{z^{s_i}}{s_i^{D_i}}\frac{z^{\widetilde{s}_i}}{\widetilde{s}_i^{\widetilde{D}_i}}\right] ds_1 d\widetilde{s}_1 ds_2 d\widetilde{s}_2$$

$$= \frac{N(D_1-1)!(\widetilde{D}_1-1)!(D_2-1)!(\widetilde{D}_2-1)!}{(\log z)^{D_1+\widetilde{D}_1+D_2+\widetilde{D}_2-4}} \times$$

$$\left\{\frac{\varphi^2 G(\mathbf{0})(\log z)^{\sum_1^2(D_i+\widetilde{D}_i-3)}}{\prod_{i=1}^2(D_i-1)!(\widetilde{D}_i-1)!}\left[\prod_{i=1}^{2}\frac{(D_i-1)(\widetilde{D}_i-1)}{(D_i+\widetilde{D}_i-3)}\right] + \mathcal{O}\Big((\log N)^{\sum_1^2(D_i+\widetilde{D}_i-3)-0.5}\Big)\right\}$$

$$= \frac{\varphi^2 G(\mathbf{0})N}{(\log z)^2}\left[\prod_{i=1}^{2}\frac{(D_i-1)(\widetilde{D}_i-1)}{(D_i+\widetilde{D}_i-3)}\right] + \mathcal{O}\left(\frac{N}{(\log N)^{2.5}}\right).$$

Since

$$G(\mathbf{0}) = \prod_{p\nmid A}\left(1-\frac{2}{p}\right)\left(1-\frac{1}{p}\right)^{-2}\prod_{p\mid A}\left(1-\frac{1}{p}\right)^{-2} = \mathfrak{S}$$

and

$$\log z = \log N - \mathsf{C}\log\log N \Rightarrow \frac{1}{\log z} = \frac{1}{\log N}\left(1+\mathcal{O}\left(\frac{\log\log N}{\log N}\right)\right),$$

the result follows.

Q.E.D.

3.3 Asymptotic formulas for Q_1 and Q_2

Before proceeding, let us first prove a lemma. Recall that we write λ_{δ_1} for $\lambda^{\mathbf{D}}_{\delta_1,\delta_2}$.

Lemma 3.3.1

Let $\lambda^{\mathbf{D}}_{\delta_1,\delta_2}$ be given by Definition 3.2.1. Suppose D_1, D_2, $\widetilde{D}_1$, $\widetilde{D}_2 \geq 2$. Then for any $\varphi \geq 4$, $(A, p_1\cdots p_r) = 1$, $p_i > Y > 6$, we have

$$\sum_{\substack{\delta_2,\widetilde{\delta}_2 \\ (\Delta_2, Ap_1\cdots p_r)=1}} \frac{\lambda_1\widetilde{\lambda}_1}{\phi(\Delta_2)} = \frac{\phi(A)}{A}\frac{\varphi\mathfrak{S}}{\log z}\frac{(D_2-1)(\widetilde{D}_2-1)}{(D_2+\widetilde{D}_2-3)} + \mathcal{O}\left(\frac{1}{Y(\log N)}\right) + \mathcal{O}\left(\frac{1}{(\log N)^{1.5}}\right),$$

where the implied constants only depend on $\mathfrak{L}$, φ, D_i, $\widetilde{D}_i$ and r.

Proof

$$\begin{aligned}
\sum_{\substack{\delta_2,\widetilde{\delta}_2\\(\Delta_2,Ap_1\cdots p_r)=1}}\frac{\lambda_1\widetilde{\lambda}_1}{\phi(\Delta_2)} &= \sum_{\substack{\delta_2,\widetilde{\delta}_2\\(\Delta_2,Ap_1\cdots p_r)=1}}\frac{1}{(\log z)^{D_2+\widetilde{D}_2-2}}\frac{\mu(\delta_2)\mu(\widetilde{\delta}_2)}{\phi(\Delta_2)}\left(\log\frac{z}{\delta_2^{\varphi}}\right)^{D_2-1}\left(\log\frac{z}{\widetilde{\delta}_2^{\varphi}}\right)^{\widetilde{D}_2-1}\\
&= \frac{(D_2-1)!(\widetilde{D}_2-1)!}{(\log z)^{D_2+\widetilde{D}_2-2}}\frac{1}{(2\pi i)^2}\int_{(1)}\int_{(1)}F_1\frac{z^{s_2}}{s_2^{D_2}}\frac{z^{\widetilde{s}_2}}{\widetilde{s}_2^{\widetilde{D}_2}}ds_2d\widetilde{s}_2,
\end{aligned}$$

where

$$\begin{aligned}
F_1 &:= \sum_{\substack{\delta_2,\widetilde{\delta}_2\\(\Delta_2,Ap_1\cdots p_r)=1}}^{\infty}\frac{\mu(\delta_2)\mu(\widetilde{\delta}_2)}{\phi(\Delta_2)}\frac{1}{(\delta_2)^{\varphi s_2}}\frac{1}{(\widetilde{\delta}_2)^{\varphi\widetilde{s}_2}}\\
&= \prod_{p\nmid Ap_1\cdots p_r}\left[1-\frac{1}{p-1}\left(\frac{1}{p^{\varphi s_2}}+\frac{1}{p^{\varphi\widetilde{s}_2}}-\frac{1}{p^{\varphi s_2+\varphi\widetilde{s}_2}}\right)\right].
\end{aligned}$$

Note that we have used Lemma 2.4.4, as $z^{1/\varphi}\notin\mathbb{N}$ and

$$\sum_{\substack{\delta_2,\widetilde{\delta}_2\\(\Delta_2,Ap_1\cdots p_r)=1}}^{\infty}\frac{|\mu(\delta_2)\mu(\widetilde{\delta}_2)|}{\phi(\Delta_2)\delta_2^{\varphi}\widetilde{\delta}_2^{\varphi}}=\prod_{p\nmid Ap_1\cdots p_r}\left[1+\frac{1}{p-1}\left(\frac{1}{p^{\varphi}}+\frac{1}{p^{\varphi}}+\frac{1}{p^{2\varphi}}\right)\right]<\infty.$$

Let

$$g_1(p) := \left(1-\frac{1}{p^{1+\varphi s_2}}\right)^{-1}\left(1-\frac{1}{p^{1+\varphi\widetilde{s}_2}}\right)^{-1}\left(1-\frac{1}{p^{1+\varphi s_2+\varphi\widetilde{s}_2}}\right),$$

so that

$$\begin{aligned}
F_1 &= \left\{F_1\cdot\prod_p g_1(p)\right\}\prod_p g(p)^{-1}\\
&:= \{G_1\}\frac{\zeta(1+\varphi s_2+\varphi\widetilde{s}_2)}{\zeta(1+\varphi s_2)\zeta(1+\varphi\widetilde{s}_2)}.
\end{aligned}$$

By Lemma 2.3.5, Proposition 2.4.2, and Lemma 3.1.1, we have

$$\begin{aligned}
&\frac{(D_2-1)!(\widetilde{D_2}-1)!}{(\log z)^{D_2+\widetilde{D_2}-2}}\frac{1}{(2\pi i)^2}\int_{(1)}\int_{(1)}F_1\frac{z^{s_2}}{s_2^{D_2}}\frac{z^{\widetilde{s}_2}}{\widetilde{s}_2^{\widetilde{D}_2}}ds_2d\widetilde{s}_2\\
=\ &\frac{(D_2-1)!(\widetilde{D_2}-1)!}{(\log z)^{D_2+\widetilde{D_2}-2}}\frac{1}{(2\pi i)^2}\int_{(1)}\int_{(1)}G_1\frac{\zeta(1+\varphi s_2+\varphi\widetilde{s}_2)}{\zeta(1+\varphi s_2)\zeta(1+\varphi\widetilde{s}_2)}\frac{z^{s_2}}{s_2^{D_2}}\frac{z^{\widetilde{s}_2}}{\widetilde{s}_2^{\widetilde{D}_2}}ds_2d\widetilde{s}_2\\
=\ &\frac{(D_2-1)!(\widetilde{D_2}-1)!}{(\log z)^{D_2+\widetilde{D_2}-2}}\times\\
&\left\{\frac{\varphi G_1(\mathbf{0})(\log z)^{D_2+\widetilde{D}_2-3}}{(D_2-1)!(\widetilde{D}_2-1)!}\frac{(D_2-1)(\widetilde{D}_2-1)}{(D_2+\widetilde{D}_2-3)}+\mathcal{O}\Big((\log N)^{D_2+\widetilde{D}_2-3-0.5}\Big)\right\}\\
=\ &\frac{\varphi G_1(\mathbf{0})}{\log z}\frac{(D_2-1)(\widetilde{D}_2-1)}{(D_2+\widetilde{D}_2-3)}+\mathcal{O}\left(\frac{1}{(\log N)^{1.5}}\right).
\end{aligned}$$

Since

$$
\begin{aligned}
G_1(\mathbf{0}) &= \prod_{p\nmid Ap_1\cdots p_r}\left(1-\frac{1}{p-1}\right)\left(1-\frac{1}{p}\right)^{-1}\prod_{p|Ap_1\cdots p_r}\left(1-\frac{1}{p}\right)^{-1} \\
&= \prod_{p\nmid A}\frac{p(p-2)}{(p-1)^2}\prod_{p|p_1\cdots p_r}\frac{(p-1)^2}{p(p-2)}\prod_{p|Ap_1\cdots p_r}\left(1-\frac{1}{p}\right)^{-1} \\
&= \prod_{p\nmid A}\left(1-\frac{1}{p}\right)^{-2}\left(1-\frac{2}{p}\right)\prod_{p|p_1\cdots p_r}\frac{(p-1)^2}{p(p-2)}\frac{p}{p-1}\prod_{p|A}\left(1-\frac{1}{p}\right)^{-1} \\
&= \left[\prod_{p\nmid A}\left(1-\frac{1}{p}\right)^{-2}\left(1-\frac{2}{p}\right)\prod_{p|A}\left(1-\frac{1}{p}\right)^{-2}\right]\prod_{p|A}\left(1-\frac{1}{p}\right)\prod_{p|p_1\cdots p_r}\frac{p-1}{p-2} \\
&= \mathfrak{S}\frac{\phi(A)}{A}\prod_{p|p_1\cdots p_r}\left(1+\mathcal{O}\left(\frac{1}{p}\right)\right),
\end{aligned}
$$

the result follows.

Q.E.D.

Theorem 3.3.2

Let $\{a_1n+b_1, a_2n+b_2\}$ be HB-admissible, $0<\eta\le\frac{1}{4}$, and let

$$
Q_1 := \sum_{N<n\le 2N}\gamma_1(a_1n+b_1)\left(\sum_{\substack{\delta_1|a_1n+b_1\\ \delta_2|a_2n+b_2}}\lambda_{\delta_1,\delta_2}^{D_1,D_2}\right)\left(\sum_{\substack{\widetilde{\delta}_1|a_1n+b_1\\ \widetilde{\delta}_2|a_2n+b_2}}\lambda_{\widetilde{\delta}_1,\widetilde{\delta}_2}^{\widetilde{D}_1,\widetilde{D}_2}\right),
$$

where $\gamma_1:\mathbb{N}\to\{0,1\}$ is defined by

$$
\gamma_1(m)=\begin{cases}1 & \text{if } m \text{ is a prime not less than } Y=z^{\eta/\varphi}\\ 0 & \text{otherwise;}\end{cases}
$$

and $\lambda_{\delta_1,\delta_2}^{\mathbf{D}}$ is given by Definition 3.2.1. Suppose D_1, D_2, $\widetilde{D}_1$, $\widetilde{D}_2\ge 2$. Then for any $\varphi\ge 4$, we have

$$
Q_1=\frac{\varphi\mathfrak{S}N}{\log^2 N}\frac{(D_2-1)(\widetilde{D}_2-1)}{(D_2+\widetilde{D}_2-3)}+\mathcal{O}\left(\frac{N}{\log^{2.5}N}\right).
$$

where the implied constant only depends on $\mathfrak{L}$, D_1, D_2, $\widetilde{D}_1$, $\widetilde{D}_2$ and φ.

Proof

We apply Theorem 2.2.2(b) with $k=2$, $j=1$. It follows that

$$\begin{aligned} Q_1 &= \sum_{a_1N<\ell\le 2a_1N} \varpi(\ell) \sum_{\substack{\delta_2,\widetilde{\delta}_2\\(\Delta_2,A\ell)=1}} \frac{[\lambda_1\widetilde{\lambda}_1]}{\phi(a_1\Delta_2)} \\ &\quad + \mathcal{O}\left(\sum_{\substack{\delta_2,\widetilde{\delta}_2\\(\Delta_2,A)=1}} |\lambda_1\widetilde{\lambda}_1| \mathcal{E}^*_\varpi(a_1N;a_1\Delta_2)\right) + \mathcal{O}\left(\sum_{\substack{\delta_1\delta_2\\ \widetilde{\delta}_1\widetilde{\delta}_2}} |\lambda\widetilde{\lambda}|\right). \end{aligned}$$

The second $\mathcal{O}$-term is $\ll z^{4/\varphi}$, which is acceptable.

The first $\mathcal{O}$-term is

$$\ll \sum_{q\le \frac{\sqrt{a_1N}}{a_1\log^{\mathsf{B}}(a_1N)}} \mu^2(q)3^{\omega(q)}\mathcal{E}^*_\varpi(a_1N;a_1q)$$

by using Lemma 2.3.1, and the observation that

$$\begin{aligned} \lambda_1\widetilde{\lambda}_1\ne 0 \;&\Rightarrow\; a_1\delta_2^2\widetilde{\delta}_2^2 \le a_1z^{4/\varphi}\le \frac{N}{\log^{2\mathsf{B}+2}N} \\ &\Rightarrow\; a_1\Delta_2^2\le \frac{N}{\log^{2\mathsf{B}+2}N} \Rightarrow a_1\Delta_2\le \frac{\sqrt{a_1N}}{\log^{\mathsf{B}}(a_1N)}. \end{aligned}$$

By Proposition 2.5.1, this $\mathcal{O}$-term is $\ll \frac{N}{\log^3 N}$, which is acceptable.

Hence, by Lemma 3.3.1 with $r=1$,

$$\begin{aligned} Q_1 &= \sum_{a_1N<\ell\le 2a_1N} \frac{\varpi(\ell)}{\phi(a_1)}\frac{\phi(A)}{A}\frac{\varphi\mathfrak{S}}{\log z}\frac{(D_2-1)(\widetilde{D}_2-1)}{(D_2+\widetilde{D}_2-3)} \\ &\quad + \mathcal{O}\left(\sum_{a_1N<\ell\le 2a_1N}\frac{\varpi(\ell)}{(\log N)^{1.5}}\right) + \mathcal{O}\left(\frac{N}{\log^{2.5}N}\right). \end{aligned}$$

It follows from the prime number theorem that

$$\begin{aligned} Q_1 &= \frac{a_1N}{\log N}\frac{1}{\phi(a_1)}\frac{\phi(A)}{A}\frac{\varphi\mathfrak{S}}{\log z}\frac{(D_2-1)(\widetilde{D}_2-1)}{(D_2+\widetilde{D}_2-3)} + \mathcal{O}\left(\frac{N}{\log^{2.5}N}\right) \\ &= \frac{\varphi\mathfrak{S}N}{\log^2 N}\frac{(D_2-1)(\widetilde{D}_2-1)}{(D_2+\widetilde{D}_2-3)} + \mathcal{O}\left(\frac{N}{\log^{2.5}N}\right), \end{aligned}$$

where in the last step we have used the fact that a_1 and A are composed of the same primes, and that $\frac{1}{\log z}=\frac{1}{\log N}+\mathcal{O}\left(\frac{1}{\log^{1.5}N}\right)$.

This completes the proof.

Q.E.D.

Theorem 3.3.3

Let $\{a_1n+b_1, a_2n+b_2\}$ be HB-admissible, $0 < \eta \le \frac{1}{4}$, and let

$$Q_2 := \sum_{N<n\le 2N} \gamma_2(a_1n+b_1) \left(\sum_{\substack{\delta_1|a_1n+b_1\\ \delta_2|a_2n+b_2}} \lambda^{D_1,D_2}_{\delta_1,\delta_2} \right) \left(\sum_{\substack{\widetilde{\delta}_1|a_1n+b_1\\ \widetilde{\delta}_2|a_2n+b_2}} \lambda^{\widetilde{D}_1,\widetilde{D}_2}_{\widetilde{\delta}_1,\widetilde{\delta}_2} \right),$$

where $\gamma_2 : \mathbb{N} \to \{0,1\}$ is defined by

$$\gamma_2(m) = \begin{cases} 1 & \text{if } m = p_1p_2,\ z^{\eta/\varphi} = Y < p_1 \le N^{1/2} < p_2 \\ 0 & \text{otherwise;} \end{cases}$$

and $\lambda^{\mathbf{D}}_{\delta_1,\delta_2}$ is given by Definition 3.2.1.

Suppose D_1, D_2, $\widetilde{D}_1$, $\widetilde{D}_2 \ge 2$. Then for any $\varphi \ge 5$, we have

$$\begin{aligned} Q_2 \;=\; & \frac{\varphi^2 \mathfrak{S} N}{(\log N)^2} \frac{(D_2-1)(\widetilde{D}_2-1)}{(D_2+\widetilde{D}_2-3)} \left(\int_\eta^{\frac{\varphi}{2}} \frac{1}{x(\varphi-x)}dx - \int_\eta^1 \frac{(1-x)^{D_1-1}}{x(\varphi-x)}dx \right. \\ & \left. - \int_\eta^1 \frac{(1-x)^{\widetilde{D}_1-1}}{x(\varphi-x)}dx + \int_\eta^1 \frac{(1-x)^{D_1+\widetilde{D}_1-2}}{x(\varphi-x)}dx \right) + \mathcal{O}\left(\frac{N(\log\log N)^{\mathbb{A}}}{\log^{2.5} N} \right), \end{aligned}$$

where the implied constant only depends on $\mathfrak{L}$, D_1, D_2, $\widetilde{D}_1$, $\widetilde{D}_2$, φ and η.

Proof

By Theorem 2.2.2(b) with $k = 2$ and $j = 2$, we have

$$\begin{aligned} Q_2 \;=\; & \sum_{r=0}^{1} \sum_{Y<p_1<\cdots<p_r} \sum_{\substack{\frac{a_1N}{p_1\cdots p_r} < \ell \le \frac{2a_1N}{p_1\cdots p_r} \\ (p_1\cdots p_r\ell, A)=1}} \gamma_2(\ell|\mathbf{p_r}) \sum_{\substack{\delta_2,\widetilde{\delta}_2 \\ (\Delta_2, Ap_1\cdots p_r\ell)=1}} \frac{\left[\sum_{[\delta_1,\widetilde{\delta}_1]=p_1\cdots p_r} \lambda_{\delta_1}\widetilde{\lambda}_{\widetilde{\delta}_1}\right]}{\phi(a_1\Delta_2)} \\ & + \mathcal{O}\left(\sum_{r=0}^{1} \sum_{p_1<\cdots<p_r} \sum_{\substack{\delta_1,\delta_2 \\ \widetilde{\delta}_1,\widetilde{\delta}_2 \\ (\Delta_1,\Delta_2)=1 \\ (\Delta_1\Delta_2,A)=1 \\ \Delta_1=p_1\cdots p_r}} |\lambda\widetilde{\lambda}| \mathcal{E}^*_{\gamma_2(\cdot|\mathbf{p_r})}\left(\frac{a_1N}{p_1\cdots p_r}; a_1\Delta_2 \right) \right) \\ & + \mathcal{O}\left(\sum_{Y<p_1} \sum_{\substack{\delta_1,\delta_2 \\ \widetilde{\delta}_1,\widetilde{\delta}_2 \\ (\Delta_i,\Delta_j)=1 \\ \Delta_1=p_1}} |\lambda\widetilde{\lambda}| \frac{N}{Yp_1}\frac{1}{\Delta_2} \right) + \mathcal{O}\left(\sum_{\substack{\delta_1,\delta_2 \\ \widetilde{\delta}_1,\widetilde{\delta}_2}} |\lambda\widetilde{\lambda}| \right), \end{aligned}$$

where the sums over $p_1, \cdots, p_r$ are restricted to those $p_1 \cdots p_r = [\delta_1, \widetilde{\delta}_1]$ with $\lambda_{\delta_1}\widetilde{\lambda}_{\widetilde{\delta}_1} \neq 0$. We take $\gamma_2(\ell|p_1) := \varpi(\ell)$. Observe that $\gamma_2(\ell|p_1) = \gamma_2(\ell p_1)$ when $p_1 > Y$, $(\ell, p_1) = 1$ and $\ell \in [\frac{N}{p_1}, \frac{2AN}{p_1}]$, because if $p_1 = [\delta_1, \widetilde{\delta}_1]$ such that $\lambda_{\delta_1}\widetilde{\lambda}_{\widetilde{\delta}_1} \neq 0$, then $p_1 = \max(\delta_1, \widetilde{\delta}_1) \leq z^{1/\varphi} < N^{1/2}$.

We go to deal with the $\mathcal{O}$-terms involved. The third $\mathcal{O}$-term is $\ll z^{4/\varphi}$, which is acceptable. By Lemma 2.3.1 and Lemma 2.3.2, the second $\mathcal{O}$-terms is

$$\ll \sum_{Y<p_1\leq z^{1/\varphi}} \frac{N}{Yp_1} \sum_{q\leq N} \frac{\mu^2(q)3^{\omega(q)}}{q} \ll \frac{N}{Y}\log^{\mathbb{A}} N,$$

which is also acceptable.

The first $\mathcal{O}$-term is

$$\ll \sum_{\substack{\delta_2,\widetilde{\delta}_2 \\ (\Delta_2,A)=1}} |\lambda_1\widetilde{\lambda}_1|\mathcal{E}^*_{\gamma_2}(a_1N; a_1\Delta_2) + \sum_{p_1\leq z^{1/\varphi}} \sum_{\substack{\delta_2,\widetilde{\delta}_2 \\ [\delta_1,\widetilde{\delta}_1]=p_1 \\ (\Delta_2,A)=1}} |\lambda_{\delta_1}\widetilde{\lambda}_{\widetilde{\delta}_1}|\mathcal{E}^*_{\varpi}\left(\frac{a_1N}{p_1}; a_1\Delta_2\right).$$

By the same argument as in the proof of Theorem 3.3.2, the first term in the last expression is

$$\ll \sum_{q\leq \frac{\sqrt{a_1N}}{a_1\log^{\mathsf{B}}(a_1N)}} \mu^2(q)3^{\omega(q)}\mathcal{E}^*_{\gamma_2}(a_1N; a_1q),$$

which is $\ll \dfrac{N}{\log^3 N}$ by Proposition 2.5.1.

On the other hand, by noting that when $p_1 = [\delta_1, \widetilde{\delta}_1] \leq z^{1/\varphi}$,

$$\begin{aligned} \lambda_{\delta_1}\widetilde{\lambda}_{\widetilde{\delta}_1} \neq 0 \quad &\Rightarrow \quad a_1p_1\delta_2^2\widetilde{\delta}_2^2 \leq a_1z^{5/\varphi} \leq \frac{N}{\log^{2\mathsf{B}+2} N} \\ &\Rightarrow \quad a_1p_1\Delta_2^2 \leq \frac{N}{\log^{2\mathsf{B}+2} N} \\ &\Rightarrow \quad a_1\Delta_2 \leq \frac{\sqrt{\frac{a_1N}{p_1}}}{\log^{\mathsf{B}}(\frac{a_1N}{p_1})}, \end{aligned}$$

the second term is

$$\ll \sum_{p_1\leq z^{1/\varphi}} \sum_{q\leq \frac{\sqrt{\frac{a_1N}{p_1}}}{a_1\log^{\mathsf{B}}(\frac{a_1N}{p_1})}} \mu^2(q)3^{\omega(q)}\mathcal{E}^*_{\varpi}\left(\frac{a_1N}{p_1}; a_1q\right).$$

By Proposition 2.5.1, this is $\ll \displaystyle\sum_{p_1\leq z^{1/\varphi}} \frac{N}{p_1\log^3 N} \ll \frac{N}{\log^{2.5} N}$.

Therefore, we have shown that all the $\mathcal{O}$-terms in the expression of Q_2 at the very beginning are acceptable.

Let's introduce a notation here: Write "$A \doteqdot B$" if $A = B + \mathcal{O}\Big(\frac{N(\log\log N)^{\mathbb{A}}}{(\log N)^{2.5}}\Big)$, where the dependence of the implied constant is the same as that stated in Theorem 3.3.3. In this way,

$$\begin{aligned} Q_2 \;\doteqdot\; & \sum_{a_1N<\ell\le 2a_1N} \frac{\gamma_2(\ell)}{\phi(a_1)} \sum_{\substack{\delta_2,\widetilde{\delta}_2\\ (\Delta_2,A\ell)=1}} \frac{[\lambda_1\widetilde{\lambda}_1]}{\phi(\Delta_2)} \\ & + \sum_{Y<p_1\le z^{1/\varphi}} \sum_{\frac{a_1N}{p_1}<\ell\le\frac{2a_1N}{p_1}} \frac{\varpi(\ell)}{\phi(a_1)} \sum_{\substack{\delta_2,\widetilde{\delta}_2\\ (\Delta_2,Ap_1\ell)=1}} \frac{[\lambda_1\widetilde{\lambda}_{p_1}+\lambda_{p_1}\widetilde{\lambda}_1+\lambda_{p_1}\widetilde{\lambda}_{p_1}]}{\phi(\Delta_2)}. \end{aligned}$$

We now use Lemma 3.3.1. Noting that

$$\begin{aligned} \sum_{\substack{\delta_2,\widetilde{\delta}_2\\ (\Delta_2,Ap_1\ell)=1}} \frac{\lambda_1\widetilde{\lambda}_{p_1}}{\phi(\Delta_2)} &= -\left(\frac{\log\dfrac{z}{p_1^{\varphi}}}{\log z}\right)^{\widetilde{D_1}-1} \sum_{\substack{\delta_2,\widetilde{\delta}_2\\ (\Delta_2,Ap_1\ell)=1}} \frac{\lambda_1\widetilde{\lambda}_1}{\phi(\Delta_2)}, \\ \sum_{\substack{\delta_2,\widetilde{\delta}_2\\ (\Delta_2,Ap_1\ell)=1}} \frac{\lambda_{p_1}\widetilde{\lambda}_{p_1}}{\phi(\Delta_2)} &= \left(\frac{\log\dfrac{z}{p_1^{\varphi}}}{\log z}\right)^{D_1+\widetilde{D_1}-2} \sum_{\substack{\delta_2,\widetilde{\delta}_2\\ (\Delta_2,Ap_1\ell)=1}} \frac{\lambda_1\widetilde{\lambda}_1}{\phi(\Delta_2)}, \end{aligned}$$

we have

$$\begin{aligned} Q_2 \;\doteqdot\; & \sum_{a_1N<\ell\le 2a_1N} \frac{\gamma_2(\ell)}{\phi(a_1)} \frac{\phi(A)}{A} \frac{\varphi\mathfrak{S}}{\log z} \frac{(D_2-1)(\widetilde{D}_2-1)}{(D_2+\widetilde{D}_2-3)} \\ & + \sum_{Y<p_1\le z^{1/\varphi}} \sum_{\frac{a_1N}{p_1}<\ell\le\frac{2a_1N}{p_1}} \frac{\varpi(\ell)}{\phi(a_1)} \frac{\phi(A)}{A} \frac{\varphi\mathfrak{S}}{\log z} \frac{(D_2-1)(\widetilde{D}_2-1)}{(D_2+\widetilde{D}_2-3)} \\ & \times\left\{-\left(\frac{\log\dfrac{z}{p_1^{\varphi}}}{\log z}\right)^{D_1-1} - \left(\frac{\log\dfrac{z}{p_1^{\varphi}}}{\log z}\right)^{\widetilde{D_1}-1} + \left(\frac{\log\dfrac{z}{p_1^{\varphi}}}{\log z}\right)^{D_1+\widetilde{D_1}-2}\right\} \\ & + \mathcal{O}\left(\sum_{a_1N<\ell\le 2a_1N} \frac{\gamma_2(\ell)}{(\log N)^{1.5}}\right) + \mathcal{O}\left(\sum_{Y<p_1\le z^{1/\varphi}} \sum_{\frac{a_1N}{p_1}<\ell\le\frac{2a_1N}{p_1}} \frac{\varpi(\ell)}{(\log N)^{1.5}}\right). \end{aligned}$$

Since

$$\sum_{a_1N<\ell\le 2a_1N} \gamma_2(\ell) = \sum_{Y<p_1\le N^{1/2}} \sum_{\frac{a_1N}{p_1}<\ell\le\frac{2a_1N}{p_1}} \varpi(\ell),$$

so by the prime number theorem, the $\mathcal{O}$-terms are

$$\ll \sum_{Y<p_1\le N^{1/2}} \frac{N}{p_1(\log N)^{2.5}} \ll \frac{N}{(\log N)^{2.5}} \log\log N,$$

which can be absorbed in $\doteqdot$.

It follows that

$$
\begin{aligned}
Q_2 \doteqdot & \sum_{Y<p_1\le N^{1/2}} \sum_{\frac{a_1N}{p_1}<\ell\le\frac{2a_1N}{p_1}} \frac{\varpi(\ell)}{\phi(a_1)}\frac{\phi(A)}{A}\frac{\varphi\mathfrak{S}}{\log z}\frac{(D_2-1)(\widetilde{D}_2-1)}{(D_2+\widetilde{D}_2-3)} \\
& + \sum_{Y<p_1\le z^{1/\varphi}} \sum_{\frac{a_1N}{p_1}<\ell\le\frac{2a_1N}{p_1}} \frac{\varpi(\ell)}{\phi(a_1)}\frac{\phi(A)}{A}\frac{\varphi\mathfrak{S}}{\log z}\frac{(D_2-1)(\widetilde{D}_2-1)}{(D_2+\widetilde{D}_2-3)} \\
& \times\left\{-\left(\frac{\log\frac{z}{p_1^\varphi}}{\log z}\right)^{D_1-1}-\left(\frac{\log\frac{z}{p_1^\varphi}}{\log z}\right)^{\widetilde{D}_1-1}+\left(\frac{\log\frac{z}{p_1^\varphi}}{\log z}\right)^{D_1+\widetilde{D}_1-2}\right\}.
\end{aligned}
$$

By prime number theorem, the right hand side is

$$
\begin{aligned}
\doteqdot & \frac{\varphi\mathfrak{S}N}{\log z}\frac{(D_2-1)(\widetilde{D}_2-1)}{(D_2+\widetilde{D}_2-3)}\sum_{Y<p_1\le N^{1/2}}\frac{1}{p_1\log(N/p_1)} \\
& + \frac{\varphi\mathfrak{S}N}{\log z}\frac{(D_2-1)(\widetilde{D}_2-1)}{(D_2+\widetilde{D}_2-3)}\sum_{Y<p_1\le z^{1/\varphi}}\frac{1}{p_1\log(N/p_1)} \\
& \times\left\{-\left(\frac{\log\frac{z}{p_1^\varphi}}{\log z}\right)^{D_1-1}-\left(\frac{\log\frac{z}{p_1^\varphi}}{\log z}\right)^{\widetilde{D}_1-1}+\left(\frac{\log\frac{z}{p_1^\varphi}}{\log z}\right)^{D_1+\widetilde{D}_1-2}\right\}.
\end{aligned}
$$

To proceed, we first expand $(\log z-\varphi\log p_1)^m$ in the braces by binomial theorem. Each term in the sums over p_1 is then of the form

$$a(\log z)^n V_1(p_1),$$

where

$$V_1(y_1)=\frac{(\log y_1)^{m_1}}{y_1\log(N/y_1)}.$$

We then apply Proposition 2.4.8. By Proposition 2.4.9, the contributions from R_i's altogether are $\ll \frac{N}{\log^{\mathbb{A}} N}$, which can be absorbed in $\doteqdot$.

The resulting V_0's are then grouped by reversing the binomial expansions. This results in

$$
\begin{aligned}
Q_2 \doteqdot & \frac{\varphi\mathfrak{S}N}{\log z}\frac{(D_2-1)(\widetilde{D}_2-1)}{(D_2+\widetilde{D}_2-3)}\left\{\int_Y^{N^{\frac12}}\frac{1}{u\log u\log\frac{N}{u}}du-\int_Y^{z^{1/\varphi}}\frac{\left(\log\frac{z}{u^\varphi}/\log z\right)^{D_1-1}}{u\log u\log\frac{N}{u}}du\right. \\
& \left.-\int_Y^{z^{1/\varphi}}\frac{\left(\log\frac{z}{u^\varphi}/\log z\right)^{\widetilde{D}_1-1}}{u\log u\log\frac{N}{u}}du+\int_Y^{z^{1/\varphi}}\frac{\left(\log\frac{z}{u^\varphi}/\log z\right)^{D_1+\widetilde{D}_1-2}}{u\log u\log\frac{N}{u}}du\right\}.
\end{aligned}
$$

We use the change of variable $u = z^{x/\varphi}$ in the integrations. As

$$\frac{du}{u\log u\log\dfrac{N}{u}} = \frac{(z^{x/\varphi})\frac{\log z}{\varphi}dx}{(z^{x/\varphi})\frac{x}{\varphi}\log z\log\dfrac{N}{z^{x/\varphi}}} = \frac{dx}{x\log\dfrac{N}{z^{x/\varphi}}},$$

this gives

$$\begin{aligned}Q_2 \ &\doteqdot\ \frac{\varphi\mathfrak{S}N}{\log z}\frac{(D_2-1)(\widetilde{D}_2-1)}{(D_2+\widetilde{D}_2-3)}\left(\int_\eta^{\frac{\varphi\log N}{2\log z}}\frac{1}{x\log\dfrac{N}{z^{x/\varphi}}}dx - \int_\eta^1\frac{(1-x)^{D_1-1}}{x\log\dfrac{N}{z^{x/\varphi}}}dx\right.\\ &\left.-\int_\eta^1\frac{(1-x)^{\widetilde{D}_1-1}}{x\log\dfrac{N}{z^{x/\varphi}}}dx + \int_\eta^1\frac{(1-x)^{D_1+\widetilde{D}_1-2}}{x\log\dfrac{N}{z^{x/\varphi}}}dx\right).\end{aligned}$$

Since $\log z = \log N - \mathsf{C}\log\log N$, so for the range of x under consideration,

$$\frac{1}{\log\dfrac{N}{z^{x/\varphi}}} = \frac{1}{(1-\frac{x}{\varphi})\log N}\cdot\frac{1}{1+\frac{x}{\varphi}\frac{\mathsf{C}\log\log N}{(1-x/\varphi)\log N}} = \frac{1}{(1-\frac{x}{\varphi})\log N}\left(1+\mathcal{O}\left(\frac{\log\log N}{\log N}\right)\right)$$

uniformly in x. Also, we have $\dfrac{\varphi\log N}{2\log z} = \dfrac{\varphi}{2} + \mathcal{O}\left(\dfrac{\log\log N}{\log N}\right)$. Therefore,

$$\begin{aligned}Q_2 \ &\doteqdot\ \frac{\varphi\mathfrak{S}N}{\log N\log z}\frac{(D_2-1)(\widetilde{D}_2-1)}{(D_2+\widetilde{D}_2-3)}\left(\int_\eta^{\frac{\varphi\log N}{2\log z}}\frac{1}{x(1-\frac{x}{\varphi})}dx - \int_\eta^1\frac{(1-x)^{D_1-1}}{x(1-\frac{x}{\varphi})}dx\right.\\ &\left.-\int_\eta^1\frac{(1-x)^{\widetilde{D}_1-1}}{x(1-\frac{x}{\varphi})}dx + \int_\eta^1\frac{(1-x)^{D_1+\widetilde{D}_1-2}}{x(1-\frac{x}{\varphi})}dx\right)\\ &\doteqdot\ \frac{\varphi^2\mathfrak{S}N}{\log N\log z}\frac{(D_2-1)(\widetilde{D}_2-1)}{(D_2+\widetilde{D}_2-3)}\left(\int_\eta^{\frac{\varphi}{2}}\frac{1}{x(\varphi-x)}dx - \int_\eta^1\frac{(1-x)^{D_1-1}}{x(\varphi-x)}dx\right.\\ &\left.-\int_\eta^1\frac{(1-x)^{\widetilde{D}_1-1}}{x(\varphi-x)}dx + \int_\eta^1\frac{(1-x)^{D_1+\widetilde{D}_1-2}}{x(\varphi-x)}dx\right)\\ &\doteqdot\ \frac{\varphi^2\mathfrak{S}N}{(\log N)^2}\frac{(D_2-1)(\widetilde{D}_2-1)}{(D_2+\widetilde{D}_2-3)}\left(\int_\eta^{\frac{\varphi}{2}}\frac{1}{x(\varphi-x)}dx - \int_\eta^1\frac{(1-x)^{D_1-1}}{x(\varphi-x)}dx\right.\\ &\left.-\int_\eta^1\frac{(1-x)^{\widetilde{D}_1-1}}{x(\varphi-x)}dx + \int_\eta^1\frac{(1-x)^{D_1+\widetilde{D}_1-2}}{x(\varphi-x)}dx\right).\end{aligned}$$

Q.E.D.

3.4 Conclusion

We now prove Theorem 1.2.1.

Take $\varphi = 5$, $\lambda^{\mathbf{D}}_{\delta_1,\delta_2}$ be that given by Definition 3.2.1, $\gamma_1 : \mathbb{N} \to \{0,1\}$ be that defined in Theorem 3.3.2, $\gamma_2 : \mathbb{N} \to \{0,1\}$ be that defined in Theorem 3.3.3. To prove the theorem, it suffices to show that for all HB-admissible 2-tuple $\{a_1n+b_1, a_2n+b_2\}$, there exist $w_{2,2}$, $w_{2,3}$, $w_{3,2}$, $w_{3,3}$ such that

$$\sum_{N<n\leq 2N} \left[\gamma_2(a_2n+b_2) \begin{matrix} \gamma_2(a_1n+b_1) \\ + \\ \gamma_1(a_1n+b_1) \end{matrix} - 1 \right] \left(\sum_{\substack{\delta_1|a_1n+b_1 \\ \delta_2|a_2n+b_2}} \begin{bmatrix} w_{2,2}\lambda^{2,2}_{\delta_1,\delta_2} & + & w_{2,3}\lambda^{2,3}_{\delta_1,\delta_2} \\ + & & + \\ w_{3,2}\lambda^{3,2}_{\delta_1,\delta_2} & + & w_{3,3}\lambda^{3,3}_{\delta_1,\delta_2} \end{bmatrix} \right)^2$$

is positive for all arbitrarily large N.

By Theorem 3.2.2, Theorem 3.3.2 and Theorem 3.3.3, it in turn suffices to show that we have $\mathbf{w}^T(V_1 + V_2 - U)\mathbf{w} > 0$, where $\mathbf{w} = (w_{2,2}, w_{2,3}, w_{3,2}, w_{3,3})$,

$$U = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \frac{4}{3} & 1 & \frac{4}{3} \\ 1 & 1 & \frac{4}{3} & \frac{4}{3} \\ 1 & \frac{4}{3} & \frac{4}{3} & \frac{16}{9} \end{pmatrix}$$ is a matrix corresponding to the result in Theorem 3.2.2,

$$V_1 = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{4}{15} & \frac{1}{5} & \frac{4}{15} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{4}{15} & \frac{1}{5} & \frac{4}{15} \end{pmatrix}$$ is a matrix corresponding to the result in Theorem 3.3.2,

and

$$V_2 = \begin{pmatrix} v^{(2,2)}_{(2,2)} + v^{(2,2)}_{(2,2)} & v^{(2,3)}_{(2,2)} + v^{(3,2)}_{(2,2)} & v^{(3,2)}_{(2,2)} + v^{(2,3)}_{(2,2)} & v^{(3,3)}_{(2,2)} + v^{(3,3)}_{(2,2)} \\ v^{(2,2)}_{(2,3)} + v^{(2,2)}_{(3,2)} & v^{(2,3)}_{(2,3)} + v^{(3,2)}_{(3,2)} & v^{(3,2)}_{(2,3)} + v^{(2,3)}_{(3,2)} & v^{(3,3)}_{(2,3)} + v^{(3,3)}_{(3,2)} \\ v^{(2,2)}_{(3,2)} + v^{(2,2)}_{(2,3)} & v^{(2,3)}_{(3,2)} + v^{(3,2)}_{(2,3)} & v^{(3,2)}_{(3,2)} + v^{(2,3)}_{(2,3)} & v^{(3,3)}_{(3,2)} + v^{(3,3)}_{(2,3)} \\ v^{(2,2)}_{(3,3)} + v^{(2,2)}_{(3,3)} & v^{(2,3)}_{(3,3)} + v^{(3,2)}_{(3,3)} & v^{(3,2)}_{(3,3)} + v^{(2,3)}_{(3,3)} & v^{(3,3)}_{(3,3)} + v^{(3,3)}_{(3,3)} \end{pmatrix},$$

where
$$v^{(\widetilde{D}_1,\widetilde{D}_2)}_{(D_1,D_2)} = \frac{(D_2-1)(\widetilde{D}_2-1)}{(D_2+\widetilde{D}_2-3)} \left(\int_\eta^{2.5} \frac{1}{x(5-x)}dx - \int_\eta^1 \frac{(1-x)^{D_1-1}}{x(5-x)}dx - \int_\eta^1 \frac{(1-x)^{\widetilde{D}_1-1}}{x(5-x)}dx + \int_\eta^1 \frac{(1-x)^{D_1+\widetilde{D}_1-2}}{x(5-x)}dx \right),$$

corresponds to the result in Theorem 3.3.3.

Since

$$\int_{\eta}^{2.5} \frac{dx}{x(5-x)} - \int_{\eta}^{1} \frac{(1-x)^{D_1-1}dx}{x(5-x)} - \int_{\eta}^{1} \frac{(1-x)^{\widetilde{D}_1-1}dx}{x(5-x)} + \int_{\eta}^{1} \frac{(1-x)^{D_1+\widetilde{D}_1-2}dx}{x(5-x)}$$

$$\begin{aligned}
&= \int_{1}^{2.5} \frac{dx}{x(5-x)} - \int_{\eta}^{1} \frac{\sum_{i=2}^{D_1-1} \mathrm{C}_i^{D_1-1}(-x)^i}{x(5-x)} dx \\
&\quad - \int_{\eta}^{1} \frac{\sum_{i=2}^{\widetilde{D}_1-1} \mathrm{C}_i^{\widetilde{D}_1-1}(-x)^i}{x(5-x)} dx + \int_{\eta}^{1} \frac{\sum_{i=2}^{D_1+\widetilde{D}_1-2} \mathrm{C}_i^{D_1+\widetilde{D}_1-2}(-x)^i}{x(5-x)} dx \\
&= \frac{\ln 4}{5} + \sum_{i=1}^{D_1-2} \mathrm{C}_{i+1}^{D_1-1} \int_{\eta}^{1} \frac{(-x)^i dx}{5-x} \\
&\quad + \sum_{i=1}^{\widetilde{D}_1-2} \mathrm{C}_{i+1}^{\widetilde{D}_1-1} \int_{\eta}^{1} \frac{(-x)^i dx}{5-x} - \sum_{i=1}^{D_1+\widetilde{D}_1-3} \mathrm{C}_{i+1}^{D_1+\widetilde{D}_1-2} \int_{\eta}^{1} \frac{(-x)^i dx}{5-x},
\end{aligned}$$

and that

$$\begin{aligned}
\int_{0}^{1} \frac{(-x)^i dx}{5-x} &= (-1)^i \left[\int_{0}^{1} \frac{5^i}{5-x} dx + \int_{0}^{1} \frac{x^i - 5^i}{5-x} dx \right] \\
&= (-1)^i \left[5^i \ln \frac{5}{4} - \int_{0}^{1} (x^{i-1} + 5x^{i-2} + \cdots + 5^{i-1}) dx \right] \\
&= (-1)^i \left[5^i \ln \frac{5}{4} - \left(\frac{1}{i} + \frac{5}{i-1} + \cdots + 5^{i-1} \right) \right],
\end{aligned}$$

so by letting $M =$

$$\begin{pmatrix}
\frac{2\ln 4}{5} + 10\ln\frac{5}{4} - \frac{14}{5} & \frac{2\ln 4}{5} - 10\ln\frac{5}{4} + \frac{17}{10} & \frac{2\ln 4}{5} - 10\ln\frac{5}{4} + \frac{17}{10} & \frac{2\ln 4}{5} - 30\ln\frac{5}{4} + \frac{31}{5} \\
\frac{2\ln 4}{5} - 10\ln\frac{5}{4} + \frac{17}{10} & \frac{7\ln 4}{15} + \frac{155}{3}\ln\frac{5}{4} - \frac{367}{30} & \frac{2\ln 4}{5} - 30\ln\frac{5}{4} + \frac{31}{5} & \frac{7\ln 4}{15} + 25\ln\frac{5}{4} - \frac{187}{30} \\
\frac{2\ln 4}{5} - 10\ln\frac{5}{4} + \frac{17}{10} & \frac{2\ln 4}{5} - 30\ln\frac{5}{4} + \frac{31}{5} & \frac{7\ln 4}{15} + \frac{155}{3}\ln\frac{5}{4} - \frac{123}{10} & \frac{7\ln 4}{15} + 25\ln\frac{5}{4} - \frac{63}{10} \\
\frac{2\ln 4}{5} - 30\ln\frac{5}{4} + \frac{31}{5} & \frac{7\ln 4}{15} + 25\ln\frac{5}{4} - \frac{187}{30} & \frac{7\ln 4}{15} + 25\ln\frac{5}{4} - \frac{63}{10} & \frac{8\ln 4}{15} + 120\ln\frac{5}{4} - \frac{416}{15}
\end{pmatrix},$$

we have

$$V_1 + V_2 - U = M + \mathcal{O}(\eta) \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Therefore, the proof is complete if we can show that for some $w_{2,2} + w_{2,3} + w_{3,2} + w_{3,3} = 1$ we have $\mathbf{w}^T M \mathbf{w} > 0$, because η can be as small as one wishes.

Suggested by the method of Lagrange multiplier, we try to solve

$$\begin{pmatrix} & & M & & -1 \\ & & & & -1 \\ & & & & -1 \\ & & & & -1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} w_{2,2} \\ w_{2,3} \\ w_{3,2} \\ w_{3,3} \\ \ell \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

By Gaussian elimination, we find that the solution is approximately equal to

$$\begin{pmatrix} w_{2,2} \\ w_{2,3} \\ w_{3,2} \\ w_{3,3} \\ \ell \end{pmatrix} = \begin{pmatrix} 0.5372 \\ 0.2568 \\ 0.1399 \\ 0.0661 \\ 0.0056 \end{pmatrix}.$$

By this choice of $w_{i,j}$, we have $\mathbf{w}^T M \mathbf{w} = \ell = 0.0056 > 0$. Q.E.D.

We end this chapter by noting that in this proof, the number of non-zero λ is about $N^{\frac{2}{5}} = N^{0.4}$, while in the proof of $\lceil 2:4\,,2 \rfloor$ in [5, Theorem 3], the number of non-zero λ is about $N^{\frac{1}{4}} = N^{0.25}$.

Chapter 4

Results due to evaluation of E_j's

This chapter aims to prove Theorem 1.2.2 and Theorem 1.2.3. The sieve used is essentially the same as that used in the literature, e.g. [5], [11]. The most novel part of the proofs is that, by involving the contribution from each E_j, we can form flexible weights in the sifting process.

4.1 Preparation

Lemma 4.1.1

Let $z = \frac{N}{\log^{\mathsf{C}} N}$, $\mathsf{C} > 0$, and let $G : \prod_{i=1}^{2} \mathbb{H}_{-a} \to \mathbb{C}$ be a bounded $\mathcal{C}^\infty$-function. Let $1 \le \mathcal{L} \le \mathcal{R} \le z$, $\varphi > 2$, and define

$$J := \frac{1}{(2\pi i)^2} \int_{(1)} \int_{(1)} G \frac{\zeta^\kappa(1+\varphi s + \varphi \widetilde{s})}{\zeta^\kappa(1+\varphi s)\zeta^\kappa(1+\varphi \widetilde{s})} \frac{\left(\frac{z}{\mathcal{L}}\right)^s}{s^D} \frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{s}}}{\widetilde{s}^{\widetilde{D}}} ds d\widetilde{s},$$

where $\kappa, D, \widetilde{D} \in \mathbb{N}$ satisfy

(i) $\kappa \ge 1$;

(ii) $D, \widetilde{D} \ge \kappa + 1$, *and*

(iii) $D, \widetilde{D} \ge \kappa + 2$ *if* $(\mathcal{L}, \mathcal{R}) \ne (1,1)$.

Then

$$\begin{aligned} J \;=\; & \frac{\varphi^\kappa G(0,0)}{(D+\widetilde{D}-\kappa-2)!} \sum_{i=0}^{D-\kappa-1} \mathrm{C}_i^{D+\widetilde{D}-\kappa-2} \Big(\log \frac{z}{\mathcal{L}}\Big)^i \Big(\log \frac{z}{\mathcal{R}}\Big)^{D+\widetilde{D}-\kappa-2-i} \mathrm{C}_{D-\kappa-1-i}^{D-2-i} (-1)^{D-\kappa-1-i} \\ & + \mathcal{O}\Big(\log^{D+\widetilde{D}-\kappa-2.5} N\Big), \end{aligned}$$

where the implied constant only depends on C, G, φ, κ, D and $\widetilde{D}$.

Proof (c.f. [7, proof of Lemma 1])

First, note that we shall make frequent use of Lemma 2.3.3 and Lemma 2.3.4 throughout the proof.

Let $T = e^{\sqrt{\log N}}$, $c_1 = \frac{c_0}{\varphi}$ and $\widetilde{c_1} = \frac{0.5c_0}{\varphi}$, where c_0 is defined in Lemma 2.3.3. We begin by shifting the s-contour to $(\frac{c_1}{\log T})$. Now (c.f. Figure 3.1.1)

$$\begin{aligned} J &= \frac{1}{(2\pi i)^2}\int_{(1)}\int_{\{\frac{c_1}{\log T}+it:\,-Y_l\le t\le Y_u\}} G\frac{\zeta^\kappa(1+\varphi s+\varphi\widetilde{s})}{\zeta^\kappa(1+\varphi s)\zeta^\kappa(1+\varphi\widetilde{s})}\frac{\left(\frac{z}{\mathcal{L}}\right)^s}{s^D}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{s}}}{\widetilde{s}^{\widetilde{D}}}ds\,d\widetilde{s} \\ &+\frac{1}{(2\pi i)^2}\int_{(1)}\int_{H_u\cup H_l} G\frac{\zeta^\kappa(1+\varphi s+\varphi\widetilde{s})}{\zeta^\kappa(1+\varphi s)\zeta^\kappa(1+\varphi\widetilde{s})}\frac{\left(\frac{z}{\mathcal{L}}\right)^s}{s^D}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{s}}}{\widetilde{s}^{\widetilde{D}}}ds\,d\widetilde{s} \\ &+\frac{1}{(2\pi i)^2}\int_{(1)}\int_{\{1+it:\,t<-Y_l \text{ or } t>Y_u\}} G\frac{\zeta^\kappa(1+\varphi s+\varphi\widetilde{s})}{\zeta^\kappa(1+\varphi s)\zeta^\kappa(1+\varphi\widetilde{s})}\frac{\left(\frac{z}{\mathcal{L}}\right)^s}{s^D}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{s}}}{\widetilde{s}^{\widetilde{D}}}ds\,d\widetilde{s}. \end{aligned}$$

Let $Y = \min\{Y_u, Y_l\}$. Observe that the second term at the right is

$$\ll \int_{(1)}\int_{H_u\cup H_l} (\log^{\mathbb{A}} T)\frac{\left(\frac{z}{\mathcal{L}}\right)^\sigma}{Y^D}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{\sigma}}}{|\widetilde{s}|^{\widetilde{D}}}|ds|\,|d\widetilde{s}| \ll \log^{\mathbb{A}} T\cdot\frac{z}{\mathcal{L}}\frac{z}{\mathcal{R}}\left(\frac{1}{Y}\right)^D,$$

and the third term at the right is

$$\ll \int_{(1)}\int_{\{1+it:|t|>Y\}} \frac{\left(\frac{z}{\mathcal{L}}\right)^\sigma}{|s|^D}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{\sigma}}}{|\widetilde{s}|^{\widetilde{D}}}|ds|\,|d\widetilde{s}| \ll \frac{z}{\mathcal{L}}\frac{z}{\mathcal{R}}\left(\frac{1}{Y}\right)^{D-1}.$$

Hence, by letting $Y_u, Y_l \to \infty$, we get

$$J = \frac{1}{(2\pi i)^2}\int_{(1)}\int_{(\frac{c_1}{\log T})} G\frac{\zeta^\kappa(1+\varphi s+\varphi\widetilde{s})}{\zeta^\kappa(1+\varphi s)\zeta^\kappa(1+\varphi\widetilde{s})}\frac{\left(\frac{z}{\mathcal{L}}\right)^s}{s^D}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{s}}}{\widetilde{s}^{\widetilde{D}}}ds\,d\widetilde{s}.$$

Similarly, we shift the $\widetilde{s}$-contour to $(\frac{\widetilde{c_1}}{\log T})$. Since

$$\begin{aligned} J &= \frac{1}{(2\pi i)^2}\int_{\{\frac{\widetilde{c_1}}{\log T}+i\widetilde{t}:-Y_l\le\widetilde{t}\le Y_u\}}\int_{(\frac{c_1}{\log T})} G\frac{\zeta^\kappa(1+\varphi s+\varphi\widetilde{s})}{\zeta^\kappa(1+\varphi s)\zeta^\kappa(1+\varphi\widetilde{s})}\frac{\left(\frac{z}{\mathcal{L}}\right)^s}{s^D}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{s}}}{\widetilde{s}^{\widetilde{D}}}ds\,d\widetilde{s} \\ &+\frac{1}{(2\pi i)^2}\int_{H_u\cup H_l}\int_{(\frac{c_1}{\log T})} G\frac{\zeta^\kappa(1+\varphi s+\varphi\widetilde{s})}{\zeta^\kappa(1+\varphi s)\zeta^\kappa(1+\varphi\widetilde{s})}\frac{\left(\frac{z}{\mathcal{L}}\right)^s}{s^D}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{s}}}{\widetilde{s}^{\widetilde{D}}}ds\,d\widetilde{s} \\ &+\frac{1}{(2\pi i)^2}\int_{\{1+i\widetilde{t}:\,\widetilde{t}<-Y_l \text{ or } \widetilde{t}>Y_u\}}\int_{(\frac{c_1}{\log T})} G\frac{\zeta^\kappa(1+\varphi s+\varphi\widetilde{s})}{\zeta^\kappa(1+\varphi s)\zeta^\kappa(1+\varphi\widetilde{s})}\frac{\left(\frac{z}{\mathcal{L}}\right)^s}{s^D}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{s}}}{\widetilde{s}^{\widetilde{D}}}ds\,d\widetilde{s}, \end{aligned}$$

and that the second and third term at the right are

$$
\begin{aligned}
&\ll \int\limits_{H_u\cup H_l}\int\limits_{(\frac{c_1}{\log T})}(\log^{\mathbb{A}}T)\frac{\frac{z}{\mathcal{L}}}{|s|^D}\frac{\frac{z}{\mathcal{R}}}{Y^{\widetilde{D}}}|ds|\,|d\widetilde{s}| + \int\limits_{\{1+i\widetilde{t}:\,|\widetilde{t}|>Y\}}\int\limits_{(\frac{c_1}{\log T})}(\log^{\mathbb{A}}T)\frac{\frac{z}{\mathcal{L}}}{|s|^D}\frac{\frac{z}{\mathcal{R}}}{|\widetilde{s}|^{\widetilde{D}}}|ds|\,|d\widetilde{s}| \\
&\ll \log^{\mathbb{A}}T\cdot\frac{z}{\mathcal{L}}\frac{z}{\mathcal{R}}\left(\frac{1}{Y}\right)^{\widetilde{D}-1},
\end{aligned}
$$

so again by letting $Y_u, Y_l \to \infty$, we get

$$
J = \frac{1}{(2\pi i)^2}\int\limits_{(\frac{\widetilde{c_1}}{\log T})}\int\limits_{(\frac{c_1}{\log T})} G\frac{\zeta^\kappa(1+\varphi s+\varphi\widetilde{s})}{\zeta^\kappa(1+\varphi s)\zeta^\kappa(1+\varphi\widetilde{s})}\frac{\left(\frac{z}{\mathcal{L}}\right)^s}{s^D}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{s}}}{\widetilde{s}^{\widetilde{D}}}ds d\widetilde{s}.
$$

Next, we go to truncate the contours to $s: |t|\le T, \widetilde{s}: |\widetilde{t}|\le 0.5T$ and denote the resulting contours by $L, \widetilde{L}$. We have

$$
\begin{aligned}
J =& \frac{1}{(2\pi i)^2}\int\limits_{\widetilde{L}}\int\limits_{L} G\frac{\zeta^\kappa(1+\varphi s+\varphi\widetilde{s})}{\zeta^\kappa(1+\varphi s)\zeta^\kappa(1+\varphi\widetilde{s})}\frac{\left(\frac{z}{\mathcal{L}}\right)^s}{s^D}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{s}}}{\widetilde{s}^{\widetilde{D}}}ds d\widetilde{s} \\
&+\mathcal{O}\left(\int\limits_{(\frac{\widetilde{c_1}}{\log T})\setminus\widetilde{L}}\int\limits_{L}\left|G\frac{\zeta^\kappa(1+\varphi s+\varphi\widetilde{s})}{\zeta^\kappa(1+\varphi s)\zeta^\kappa(1+\varphi\widetilde{s})}\frac{\left(\frac{z}{\mathcal{L}}\right)^s}{s^D}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{s}}}{\widetilde{s}^{\widetilde{D}}}\right||ds|\,|d\widetilde{s}|\right) \\
&+\mathcal{O}\left(\int\limits_{(\frac{\widetilde{c_1}}{\log T})}\int\limits_{(\frac{c_1}{\log T})\setminus L}\left|G\frac{\zeta^\kappa(1+\varphi s+\varphi\widetilde{s})}{\zeta^\kappa(1+\varphi s)\zeta^\kappa(1+\varphi\widetilde{s})}\frac{\left(\frac{z}{\mathcal{L}}\right)^s}{s^D}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{s}}}{\widetilde{s}^{\widetilde{D}}}\right||ds|\,|d\widetilde{s}|\right).
\end{aligned}
$$

The $\mathcal{O}$-terms here are

$$
\ll \int\limits_{(\frac{\widetilde{c_1}}{\log T})\setminus\widetilde{L}}\int\limits_{L}(\log^{\mathbb{A}}T)\frac{z^{\frac{c}{\log T}}}{|s|^D|\widetilde{s}|^{\widetilde{D}}} + \int\limits_{(\frac{\widetilde{c_1}}{\log T})}\int\limits_{(\frac{c_1}{\log T})\setminus L}(\log^{\mathbb{A}}T)\frac{z^{\frac{c}{\log T}}}{|s|^D|\widetilde{s}|^{\widetilde{D}}} \ll \frac{\log^{\mathbb{A}}T z^{\frac{c}{\log T}}}{T}.
$$

It follows that

(4.1)

$$
J = \frac{1}{(2\pi i)^2}\iint\limits_{\widetilde{L}\,L} G\frac{\zeta^\kappa(1+\varphi s+\varphi\widetilde{s})}{\zeta^\kappa(1+\varphi s)\zeta^\kappa(1+\varphi\widetilde{s})}\frac{\left(\frac{z}{\mathcal{L}}\right)^s}{s^D}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{s}}}{\widetilde{s}^{\widetilde{D}}}ds d\widetilde{s} + \mathcal{O}\left(\frac{\log^{\mathbb{A}}T z^{\frac{c}{\log T}}}{T}\right).
$$

We then shift L to $L^-: \dfrac{-c_0/\varphi}{\log T}+it, |t|\le T$. (c.f. Figure 3.1.2) For each $\widetilde{s}\in\widetilde{L}$,

$$\begin{aligned}
&\frac{1}{2\pi i}\int\limits_{L} G\frac{\zeta^\kappa(1+\varphi s+\varphi\widetilde{s})}{\zeta^\kappa(1+\varphi s)\zeta^\kappa(1+\varphi\widetilde{s})}\frac{\left(\frac{z}{\mathcal{L}}\right)^s}{s^D}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{s}}}{\widetilde{s}^{\widetilde{D}}}ds\\
&\quad= \frac{1}{2\pi i}\int\limits_{L^-} G\frac{\zeta^\kappa(1+\varphi s+\varphi\widetilde{s})}{\zeta^\kappa(1+\varphi s)\zeta^\kappa(1+\varphi\widetilde{s})}\frac{\left(\frac{z}{\mathcal{L}}\right)^s}{s^D}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{s}}}{\widetilde{s}^{\widetilde{D}}}ds\\
&\qquad+\frac{1}{2\pi i}\int\limits_{H} G\frac{\zeta^\kappa(1+\varphi s+\varphi\widetilde{s})}{\zeta^\kappa(1+\varphi s)\zeta^\kappa(1+\varphi\widetilde{s})}\frac{\left(\frac{z}{\mathcal{L}}\right)^s}{s^D}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{s}}}{\widetilde{s}^{\widetilde{D}}}ds\\
&\qquad+\left\{\operatorname*{Res}_{s=0}+\operatorname*{Res}_{s=-\widetilde{s}}\right\},
\end{aligned}$$

where H denotes the upper and lower horizontal contours between L and L^-. Integrating the right hand side over $\widetilde{L}$, the second term is

$$\ll \int\limits_{\widetilde{L}}\int\limits_{H}(\log^{\mathbb{A}}T)\frac{\left(\frac{z}{\mathcal{L}}\right)^\sigma}{|s|^D}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{\sigma}}}{|\widetilde{s}|^{\widetilde{D}}}|ds|\,|d\widetilde{s}| \ll \frac{\log^{\mathbb{A}}T z^{\frac{c}{\log T}}}{T},$$

and the first term is

$$\begin{aligned}
&\ll \int\limits_{\widetilde{L}}\int\limits_{L^-}\frac{|\zeta^\kappa(1+\varphi s+\varphi\widetilde{s})|}{|\varphi s|^\kappa|\zeta^\kappa(1+\varphi s)|\,|\varphi\widetilde{s}|^\kappa|\zeta^\kappa(1+\varphi\widetilde{s})|}\frac{\left(\frac{z}{\mathcal{L}}\right)^\sigma}{|s|^{D-\kappa}}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{\sigma}}}{|\widetilde{s}|^{\widetilde{D}-\kappa}}|ds|\,|d\widetilde{s}|\\
&\ll \int\limits_{\widetilde{L}}\int\limits_{L^-}(\log^\kappa T)\frac{\left(\frac{z}{\mathcal{R}}\right)^{\frac{-c_0/\varphi}{\log T}}}{|s|^{D-\kappa}}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\frac{0.5c_0/\varphi}{\log T}}}{|\widetilde{s}|^{\widetilde{D}-\kappa}}|ds|\,|d\widetilde{s}|.
\end{aligned}$$

If $(\mathcal{L},\mathcal{R})=(1,1)$, this is $\ll \log^{\mathbb{A}}T z^{-\frac{c}{\log T}}$; else, by $D,\widetilde{D}\ge\kappa+2$, this is

$$\ll \int\limits_{\widetilde{L}}\int\limits_{L^-}\frac{\log^\kappa T}{|s|^{D-\kappa}|\widetilde{s}|^{\widetilde{D}-\kappa}}|ds|\,|d\widetilde{s}| \ll (\log T)^{\kappa+D-\kappa-1+\widetilde{D}-\kappa-1}=(\log N)^{\frac{D+\widetilde{D}-\kappa-2}{2}}.$$

Since

$$\log^{\mathbb{A}}T z^{-\frac{c}{\log T}} \ll \log^{\mathbb{A}}N\,e^{-c\sqrt{\log N}} \ll (\log N)^{\frac{D+\widetilde{D}-\kappa-2}{2}} \ll (\log N)^{\frac{D+\widetilde{D}-3}{2}},$$

this term is always $\ll (\log N)^{\frac{D+\widetilde{D}-3}{2}}$, so (4.1) implies

$$J = \frac{1}{2\pi i}\int\limits_{\widetilde{L}}\left\{\operatorname*{Res}_{s=0}+\operatorname*{Res}_{s=-\widetilde{s}}\right\}d\widetilde{s}+\mathcal{O}\left(\frac{\log^{\mathbb{A}}T z^{\frac{c}{\log T}}}{T}+(\log N)^{\frac{D+\widetilde{D}-3}{2}}\right). \tag{4.2}$$

For each $\widetilde{s} \in \widetilde{L}$, let $C = C(\widetilde{s})$ be the s-circle defined by $C : |s + \widetilde{s}| = \frac{1}{\log^2 T}$ described in the positive sense. By noting C does not enclose the point 0, we see that

$$\begin{aligned}
\int\limits_{\widetilde{L}} \left\{\underset{s=-\widetilde{s}}{\mathrm{Res}}\right\} d\widetilde{s} &= \frac{1}{2\pi i}\int\limits_{\widetilde{L}}\int\limits_{C} G\frac{\zeta^{\kappa}(1+\varphi s+\varphi\widetilde{s})}{\zeta^{\kappa}(1+\varphi s)\zeta^{\kappa}(1+\varphi\widetilde{s})}\frac{\left(\frac{z}{\mathcal{L}}\right)^{s}}{s^{D}}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{s}}}{\widetilde{s}^{\widetilde{D}}}dsd\widetilde{s} \\
&\ll \int\limits_{\widetilde{L}}\int\limits_{C}\frac{|\zeta^{\kappa}(1+\varphi s+\varphi\widetilde{s})|}{|\varphi s|^{\kappa}|\zeta^{\kappa}(1+\varphi s)|\,|\varphi\widetilde{s}|^{\kappa}|\zeta^{\kappa}(1+\varphi\widetilde{s})|}\frac{\left(\frac{z}{\mathcal{L}}\right)^{\sigma}}{|s|^{D-\kappa}}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{\sigma}}}{|\widetilde{s}|^{\widetilde{D}-\kappa}}|ds|\,|d\widetilde{s}| \\
&\ll \int\limits_{\widetilde{L}}\int\limits_{C}(\log^{2\kappa}T)\frac{\left(\frac{z}{\mathcal{L}}\right)^{\sigma}}{|s|^{D-\kappa}}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{\sigma}}}{|\widetilde{s}|^{\widetilde{D}-\kappa}}|ds|\,|d\widetilde{s}|.
\end{aligned}$$

For $\widetilde{s} \in \widetilde{L}$, $s \in C$, we have $|s| \geq |\widetilde{s}| - |s+\widetilde{s}| \gg |\widetilde{s}|$. Also, by $|\sigma+\widetilde{\sigma}| \leq |s+\widetilde{s}| = \frac{1}{\log^2 T}$, we have $\sigma \leq -\widetilde{\sigma} + |\sigma+\widetilde{\sigma}| \leq -\widetilde{\sigma} + \frac{1}{\log^2 T}$, thus

$$\left(\frac{z}{\mathcal{L}}\right)^{\sigma}\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{\sigma}} \leq \left(\frac{z}{\mathcal{L}}\right)^{-\widetilde{\sigma}} z^{\frac{1}{\log^2 T}}\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{\sigma}} \leq \left(\frac{z}{\mathcal{R}}\right)^{-\widetilde{\sigma}} z^{\frac{1}{\log N}}\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{\sigma}} \ll 1.$$

Therefore,

$$\int\limits_{\widetilde{L}} \left\{\underset{s=-\widetilde{s}}{\mathrm{Res}}\right\} d\widetilde{s} \ll \int\limits_{\widetilde{L}}\int\limits_{C}\frac{\log^{2\kappa}T}{|\widetilde{s}|^{D+\widetilde{D}-2\kappa}} \ll (\log T)^{2\kappa-2+D+\widetilde{D}-2\kappa-1} = (\log T)^{D+\widetilde{D}-3}.$$

Inserting this into (4.2), we obtain

$$J = \frac{1}{2\pi i}\int\limits_{\widetilde{L}}\left\{\underset{s=0}{\mathrm{Res}}\right\} d\widetilde{s} + \mathcal{O}\left(\frac{\log^{\mathbb{A}}T z^{\frac{c}{\log T}}}{T} + (\log N)^{\frac{D+\widetilde{D}-3}{2}}\right). \tag{4.3}$$

Lastly, we shift $\widetilde{L}$ to $\widetilde{L}^{-} : \dfrac{-c_0/\varphi}{\log T} + i\widetilde{t},\ |\widetilde{t}| \leq 0.5T$. By Lemma 2.4.3, we have

$$\frac{1}{2\pi i}\int\limits_{\widetilde{L}}\left\{\underset{s=0}{\mathrm{Res}}\right\} d\widetilde{s} = \frac{1}{2\pi i}\int\limits_{\widetilde{L}^{-}\cup H}\left\{\underset{s=0}{\mathrm{Res}}\right\} d\widetilde{s} + \left\{\underset{\widetilde{s}=0}{\mathrm{Res}}\left\{\underset{s=0}{\mathrm{Res}}\right\}\right\},$$

where again H denotes the horizontal contours involved. Let C be the s-circle defined by $C : |s| = \frac{1}{\log^2 T}$. By noting C does not enclose the point $-\widetilde{s}$ for all $\widetilde{s} \in \widetilde{L}^{-} \cup H$, we see that

$$\begin{aligned}
\int\limits_{\widetilde{L}^{-}\cup H}\left\{\underset{s=0}{\mathrm{Res}}\right\} d\widetilde{s} &= \frac{1}{2\pi i}\int\limits_{\widetilde{L}^{-}}\int\limits_{C} G\frac{\zeta^{\kappa}(1+\varphi s+\varphi\widetilde{s})}{\zeta^{\kappa}(1+\varphi s)\zeta^{\kappa}(1+\varphi\widetilde{s})}\frac{\left(\frac{z}{\mathcal{L}}\right)^{s}}{s^{D}}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{s}}}{\widetilde{s}^{\widetilde{D}}}dsd\widetilde{s} \\
&\quad + \frac{1}{2\pi i}\int\limits_{H}\int\limits_{C} G\frac{\zeta^{\kappa}(1+\varphi s+\varphi\widetilde{s})}{\zeta^{\kappa}(1+\varphi s)\zeta^{\kappa}(1+\varphi\widetilde{s})}\frac{\left(\frac{z}{\mathcal{L}}\right)^{s}}{s^{D}}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{s}}}{\widetilde{s}^{\widetilde{D}}}dsd\widetilde{s}.
\end{aligned}$$

The second term at the right is

$$\ll \int_H \int_C (\log^\kappa T) \frac{1}{|s|^{D-\kappa}} \frac{z^{\frac{c}{\log T}}}{|\widetilde{s}|^{\widetilde{D}-\kappa}} |ds|\,|d\widetilde{s}| \ll \frac{\log^{\mathbb{A}} T z^{\frac{c}{\log T}}}{T},$$

and the first term at the right is

$$\ll \int_{\widetilde{L}^-} \int_C (\log^\kappa T) \frac{\left(\frac{z}{\mathcal{L}}\right)^\sigma}{|s|^{D-\kappa}} \frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{\sigma}}}{|\widetilde{s}|^{\widetilde{D}-\kappa}} |ds|\,|d\widetilde{s}|.$$

If $(\mathcal{L}, \mathcal{R}) = (1, 1)$, this is $\ll \log^{\mathbb{A}} T z^{-\frac{c}{\log T}}$; else, by $D, \widetilde{D} \geq \kappa + 2$, this is

$$\ll \int_{\widetilde{L}^-} \int_C (\log^\kappa T) \frac{1}{|s|^{D-\kappa}} \frac{1}{|\widetilde{s}|^{\widetilde{D}-\kappa}} |ds|\,|d\widetilde{s}| \ll (\log T)^{\kappa-2+2D-2\kappa+\widetilde{D}-\kappa-1}.$$

Therefore, (4.3) yields

$$J = \left\{ \operatorname*{Res}_{\widetilde{s}=0} \left\{ \operatorname*{Res}_{s=0} \right\} \right\} + \mathcal{O}\left(\frac{\log^{\mathbb{A}} T z^{\frac{c}{\log T}}}{T} + (\log N)^{\frac{D+\widetilde{D}-3}{2}} + (\log N)^{\frac{2D+\widetilde{D}-2\kappa-3}{2}} \right).$$

The $\mathcal{O}$-term here is

$$\begin{aligned} &\ll \frac{\log^{\mathbb{A}} N\, e^{c\sqrt{\log N}}}{e^{\sqrt{\log N}}} + (\log N)^{D+\widetilde{D}-\kappa-2.5-\frac{D+\widetilde{D}-2\kappa-2}{2}} + (\log N)^{D+\widetilde{D}-\kappa-2.5-\frac{\widetilde{D}-2}{2}} \\ &\ll (\log N)^{D+\widetilde{D}-\kappa-2.5} \qquad (\text{ by } D, \widetilde{D} \geq \kappa + 1), \end{aligned}$$

which is acceptable. It remains to deal with the iterated residue.

Let $\widetilde{C}$ be the $\widetilde{s}$-circle $|\widetilde{s}| = 2r$, C be the s-circle $|s| = r$, where r is a suitably small number independent of $z, \mathcal{L}, \mathcal{R}$. Then

$$\begin{aligned} \left\{ \operatorname*{Res}_{\widetilde{s}=0} \left\{ \operatorname*{Res}_{s=0} \right\} \right\} &= \frac{1}{2\pi i} \int_{\widetilde{C}} \left\{ \operatorname*{Res}_{s=0} \right\} d\widetilde{s} \\ &= \frac{1}{(2\pi i)^2} \int_{\widetilde{C}} \int_C G \frac{\zeta^\kappa(1+\varphi s+\varphi\widetilde{s})}{\zeta^\kappa(1+\varphi s)\zeta^\kappa(1+\varphi\widetilde{s})} \frac{\left(\frac{z}{\mathcal{L}}\right)^s}{s^D} \frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{s}}}{\widetilde{s}^{\widetilde{D}}} ds d\widetilde{s}, \end{aligned}$$

where in the last step we use the fact that for each $\widetilde{s} \in \widetilde{C}$, if $|s| \leq r$, then $|s + \widetilde{s}| \geq |\widetilde{s}| - |s| \geq 2r - r > 0$, so C does not enclose the pole $s = -\widetilde{s}$.

Let

$$Z(s,\widetilde{s}) := G(s,\widetilde{s})\frac{(\varphi s+\varphi\widetilde{s})^{\kappa}\zeta^{\kappa}(1+\varphi s+\varphi\widetilde{s})}{(\varphi s)^{\kappa}\zeta^{\kappa}(1+\varphi s)(\varphi\widetilde{s})^{\kappa}\zeta^{\kappa}(1+\varphi\widetilde{s})},$$

so that

$$\Big\{\operatorname*{Res}_{\widetilde{s}=0}\Big\{\operatorname*{Res}_{s=0}\Big\}\Big\} = \frac{1}{(2\pi i)^2}\int\limits_{\widetilde{C}}\int\limits_{C}\frac{(\varphi s)^{\kappa}(\varphi\widetilde{s})^{\kappa}}{(\varphi s+\varphi\widetilde{s})^{\kappa}}Z(s,\widetilde{s})\frac{\left(\frac{z}{\mathcal{L}}\right)^{s}}{s^{D}}\frac{\left(\frac{z}{\mathcal{R}}\right)^{\widetilde{s}}}{\widetilde{s}^{\widetilde{D}}}ds d\widetilde{s}.$$

By Lemma 2.4.5, the integral equals

$$\frac{1}{(2\pi i)^2}\int\limits_{\widetilde{C}}\int\limits_{C'}\frac{\varphi^{\kappa}\xi^{\kappa}}{(1+\xi)^{\kappa}}\frac{Z(\widetilde{s}\xi,\widetilde{s})}{\widetilde{s}^{D+\widetilde{D}-\kappa-1}}\frac{e^{\widetilde{s}(\xi\log\frac{z}{\mathcal{L}}+\log\frac{z}{\mathcal{R}})}}{\xi^{D}}d\xi d\widetilde{s},$$

where C' is the circle $|\xi|=1/2$.

For each $\xi\in C'$, $Z(\widetilde{s}\xi,\widetilde{s})$ is an analytic function in $\widetilde{s}$ for $|\widetilde{s}|<2r$, and uniformly in ξ we have $Z(\widetilde{s}\xi,\widetilde{s})\ll 1$ on this domain. Hence, for any fixed $n\in\mathbb{N}$, by Cauchy's estimation, uniformly in ξ we have $\frac{\partial^n}{\partial\widetilde{s}^n}Z(\widetilde{s}\xi,\widetilde{s})\big|_{\widetilde{s}=0}\ll r^{-n}\ll 1$ as well.

Therefore, by changing the order of integration,

$$\begin{aligned}\Big\{\operatorname*{Res}_{\widetilde{s}=0}\Big\{\operatorname*{Res}_{s=0}\Big\}\Big\} &= \frac{1}{(2\pi i)^2}\int\limits_{C'}\int\limits_{\widetilde{C}}\frac{\varphi^{\kappa}\xi^{\kappa}}{(1+\xi)^{\kappa}}\frac{Z(\widetilde{s}\xi,\widetilde{s})}{\widetilde{s}^{D+\widetilde{D}-\kappa-1}}\frac{e^{\widetilde{s}(\xi\log\frac{z}{\mathcal{L}}+\log\frac{z}{\mathcal{R}})}}{\xi^{D}}d\widetilde{s}d\xi\\ &= \frac{\varphi^{\kappa}Z(0,0)}{(D+\widetilde{D}-\kappa-2)!}\frac{1}{2\pi i}\int\limits_{C'}\frac{(\xi\log\frac{z}{\mathcal{L}}+\log\frac{z}{\mathcal{R}})^{D+\widetilde{D}-\kappa-2}}{(1+\xi)^{\kappa}\xi^{D-\kappa}}d\xi\\ &\quad+\mathcal{O}\left(\int\limits_{C'}\frac{(|\xi|+1)^{D+\widetilde{D}-\kappa-3}(\log z)^{D+\widetilde{D}-\kappa-3}}{(1-|\xi|)^{\kappa}|\xi|^{D-\kappa}}|d\xi|\right).\end{aligned}$$

The $\mathcal{O}$-term here is $\ll\log^{D+\widetilde{D}-\kappa-3}N$, which is acceptable. The main term is, by $Z(0,0)=G(0,0)$ and $\frac{1}{(1-s)^n}=\sum_{r=0}^{\infty}\mathrm{C}_r^{n+r-1}s^r$,

$$=\frac{\varphi^{\kappa}G(0,0)}{(D+\widetilde{D}-\kappa-2)!}\left[\sum_{i=0}^{D-\kappa-1}\mathrm{C}_i^{D+\widetilde{D}-\kappa-2}\left(\log\frac{z}{\mathcal{L}}\right)^{i}\left(\log\frac{z}{\mathcal{R}}\right)^{D+\widetilde{D}-\kappa-2-i}\mathrm{C}_{D-\kappa-1-i}^{D-2-i}(-1)^{D-\kappa-1-i}\right].$$

Q.E.D.

We end this section by the following identity:

$$\sum_{i=0}^{D-\kappa-1}\mathrm{C}_i^{D+\widetilde{D}-\kappa-2}\mathrm{C}_{D-\kappa-1-i}^{D-2-i}(-1)^{D-\kappa-1-i}=\mathrm{C}_{D-\kappa-1}^{D+\widetilde{D}-2\kappa-2}, \tag{4.4}$$

which can be seen by noting that the left hand side is the coefficient of $x^{D-\kappa-1}$ in $\frac{(1+x)^{D+\widetilde{D}-\kappa-2}}{(1+x)^{\kappa}}$. It will be useful later.

4.2 The choice of λ and the asymptotic formula for Q_0

Let $D \in \mathbb{N}$ and $\varphi > 2$. We shall make use of the following λ for general k:

Defintion 4.2.1

$$\lambda^{D}_{\delta_1,\ldots,\delta_k} := \begin{cases} \mu(\delta_1)\cdots\mu(\delta_k)\left(\dfrac{\log\dfrac{z}{\delta_1^{\varphi}\cdots\delta_k^{\varphi}}}{\log z}\right)^{D-1} & \text{if } \delta_1^{\varphi}\cdots\delta_k^{\varphi} \le z := \frac{N}{\log^{\mathsf{C}} N}, \\ 0 & \text{otherwise.} \end{cases}$$

where $\mathsf{C} = 2\mathsf{B}(3k,3k)+3$, and $\mathsf{B}(3k,3k)$ is the one given by Proposition 2.5.1.

By varying N if necessary, throughout this chapter we assume that $z^{1/\varphi} \notin \mathbb{N}$. Also, recall Definition 2.1.2 and the discussion of $\mathfrak{S}$ right after it.

Theorem 4.2.2

Let $\mathfrak{L}$ be HB-admissible, and let

$$Q_0 := \sum_{N<n\le 2N}\left(\sum_{\delta_i|a_in+b_i}\lambda^{D}_{\delta_1,\ldots,\delta_k}\right)\left(\sum_{\widetilde{\delta}_i|a_in+b_i}\lambda^{\widetilde{D}}_{\widetilde{\delta}_1,\ldots,\widetilde{\delta}_k}\right),$$

where $\lambda^{D}_{\delta_1,\ldots,\delta_k}$ is given by Defintion 4.2.1. Suppose D, $\widetilde{D} \ge k+1$. Then for any $\varphi > 2$, we have

$$Q_0 = \frac{\varphi^k\mathfrak{S}N}{\log^k N}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-2)!}\mathrm{C}^{D+\widetilde{D}-2k-2}_{D-k-1} + \mathcal{O}\left(\frac{N}{\log^{k+0.5}N}\right),$$

where the implied only depends on $\mathfrak{L}$, D, $\widetilde{D}$ and φ.

Proof

By Theorem 2.2.2(a), we have

$$Q_0 = \sum_{\substack{\delta_1,\ldots,\delta_k\\ \widetilde{\delta}_1,\ldots,\widetilde{\delta}_k\\ (\Delta_i,\Delta_j)=1\\ (\Delta_1\cdots\Delta_k,A)=1}}\lambda\widetilde{\lambda}\frac{N}{\Delta_1\cdots\Delta_k} + \mathcal{O}\left(\sum_{\substack{\delta_1,\ldots,\delta_k\\ \widetilde{\delta}_1,\ldots,\widetilde{\delta}_k}} 1\right).$$

The $\mathcal{O}$-term is

$$\ll \sum_{\substack{\delta_1\le z^{1/\varphi}\\ \widetilde{\delta}_1\le z^{1/\varphi}}}\ \sum_{\substack{\delta_2\le z^{1/\varphi}/\delta_1\\ \widetilde{\delta}_2\le z^{1/\varphi}/\widetilde{\delta}_1}}\cdots\sum_{\substack{\delta_k\le z^{1/\varphi}/(\delta_1\cdots\delta_{k-1})\\ \widetilde{\delta}_k\le z^{1/\varphi}/(\widetilde{\delta}_1\cdots\widetilde{\delta}_{k-1})}} 1 \ll z^{\varepsilon+2/\varphi},$$

which is acceptable.

The main term is

$$\begin{aligned}
&= \frac{N}{\log^{D+\widetilde{D}-2} z} \sum_{\substack{\delta_1,\ldots,\delta_k \\ \widetilde{\delta}_1,\ldots,\widetilde{\delta}_k \\ (\Delta_i,\Delta_j)=1 \\ (\Delta_1\cdots\Delta_k,A)=1}} \frac{[\mu(\delta_1)\cdots\mu(\delta_k)][\mu(\widetilde{\delta}_1)\cdots\mu(\widetilde{\delta}_k)]}{\Delta_1\cdots\Delta_k} \left(\log\frac{z}{\delta_1^{\varphi}\cdots\delta_k^{\varphi}}\right)^{D-1} \left(\log\frac{z}{\widetilde{\delta}_1^{\varphi}\cdots\widetilde{\delta}_k^{\varphi}}\right)^{\widetilde{D}-1} \\
&= \frac{N(D-1)!(\widetilde{D}-1)!}{\log^{D+\widetilde{D}-2} z} \frac{1}{(2\pi i)^2} \int_{(1)}\int_{(1)} F(s,\widetilde{s}) \frac{z^s}{s^D} \frac{z^{\widetilde{s}}}{\widetilde{s}^{\widetilde{D}}} ds d\widetilde{s},
\end{aligned}$$

where

$$\begin{aligned}
F &:= \sum_{\substack{\delta_1,\ldots,\delta_k \\ \widetilde{\delta}_1,\ldots,\widetilde{\delta}_k \\ (\Delta_i,\Delta_j)=1 \\ (\Delta_1\cdots\Delta_k,A)=1}}^{\infty} \frac{[\mu(\delta_1)\cdots\mu(\delta_k)][\mu(\widetilde{\delta}_1)\cdots\mu(\widetilde{\delta}_k)]}{\Delta_1\cdots\Delta_k} \frac{1}{(\delta_1^{\varphi}\cdots\delta_k^{\varphi})^s} \frac{1}{(\widetilde{\delta}_1^{\varphi}\cdots\widetilde{\delta}_k^{\varphi})^{\widetilde{s}}} \\
&= \prod_{p\nmid A} \left[1 - \frac{1}{p}\left(\frac{k}{p^{\varphi s}} + \frac{k}{p^{\varphi\widetilde{s}}} - \frac{k}{p^{\varphi s+\varphi\widetilde{s}}}\right)\right].
\end{aligned}$$

Note that we have used Lemma 2.4.4, as $z^{1/\varphi} \notin \mathbb{N}$ and

$$\sum_{\substack{\delta_1,\ldots,\delta_k \\ \widetilde{\delta}_1,\ldots,\widetilde{\delta}_k \\ (\Delta_i,\Delta_j)=1 \\ (\Delta_1\cdots\Delta_k,A)=1}}^{\infty} \frac{|\mu(\delta_1)\cdots\mu(\delta_k)\mu(\widetilde{\delta}_1)\cdots\mu(\widetilde{\delta}_k)|}{\Delta_1\cdots\Delta_k\cdot\delta_1^{\varphi}\cdots\delta_k^{\varphi}\widetilde{\delta}_1^{\varphi}\cdots\widetilde{\delta}_k^{\varphi}} = \prod_{p\nmid A}\left[1+\frac{1}{p}\left(\frac{k}{p^{\varphi}}+\frac{k}{p^{\varphi}}+\frac{k}{p^{2\varphi}}\right)\right] < \infty.$$

Let

$$g(p) := \left(1-\frac{1}{p^{1+\varphi s}}\right)^{-k}\left(1-\frac{1}{p^{1+\varphi\widetilde{s}}}\right)^{-k}\left(1-\frac{1}{p^{1+\varphi s+\varphi\widetilde{s}}}\right)^{k},$$

so that

$$\begin{aligned}
F &= \left\{F\cdot\prod_p g(p)\right\}\prod_p g(p)^{-1} = \left\{F\cdot\prod_p g(p)\right\}\frac{\zeta^k(1+\varphi s+\varphi\widetilde{s})}{\zeta^k(1+\varphi s)\zeta^k(1+\varphi\widetilde{s})} \\
&:= \{G(s,\widetilde{s})\}\frac{\zeta^k(1+\varphi s+\varphi\widetilde{s})}{\zeta^k(1+\varphi s)\zeta^k(1+\varphi\widetilde{s})}.
\end{aligned}$$

By Lemma 2.3.5, Proposition 2.4.2, and Lemma 4.1.1 ($\kappa = k$, $\mathcal{L} = \mathcal{R} = 1$),

$$\begin{aligned}
Q_0 = {} & \frac{N(D-1)!(\widetilde{D}-1)!}{\log^{D+\widetilde{D}-2} z}\frac{\varphi^k G(0,0)}{(D+\widetilde{D}-k-2)!}\sum_{i=0}^{D-k-1} \mathrm{C}_i^{D+\widetilde{D}-k-2}(\log z)^{D+\widetilde{D}-k-2}\mathrm{C}_{D-k-1-i}^{D-2-i}(-1)^{D-k-1-i} \\
& + \mathcal{O}\left(\frac{N}{\log^{k+0.5} N}\right).
\end{aligned}$$

Since

$$G(0,0)=\prod_{p\nmid A}\left(1-\frac{k}{p}\right)\left(1-\frac{1}{p}\right)^{-k}\prod_{p|A}\left(1-\frac{1}{p}\right)^{-k}=\mathfrak{S},$$

so together with (4.4), we have

$$Q_0=\frac{\varphi^k\mathfrak{S}N}{\log^k z}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-2)!}\mathrm{C}_{D-k-1}^{D+\tilde{D}-2k-2}+\mathcal{O}\left(\frac{N}{\log^{k+0.5}N}\right).$$

Noting that

$$\log z=\log N-\mathsf{C}\log\log N\Rightarrow\frac{1}{\log z}=\frac{1}{\log N}\left(1+\mathcal{O}\left(\frac{\log\log N}{\log N}\right)\right),$$

the result follows.

Q.E.D.

4.3 Towards $j\geq 1$

We proceed to find the asymptotic formula for Q_j when $j\geq 1$. We start by fixing our E_j-number-counting functions.

Definition 4.3.1

Let $0<\eta\leq 1-\frac{2}{\varphi}$ *and take* $Y:=z^{\eta/\varphi}$. *For any* $j\geq 1$, *define*

$$\gamma_j(m)=\begin{cases}1 & \text{if } m=p_1\cdots p_j,\ Y<p_1<\cdots<p_j,\\ 0 & \text{otherwise.}\end{cases}$$

We go to use Theorem 2.2.2(b). Given $p_1\cdots p_r=[\delta_1,\widetilde{\delta}_1]$ such that $\lambda_{\delta_1}\widetilde{\lambda}_{\widetilde{\delta}_1}\neq 0$, we take $\gamma_j(\ell|\mathbf{p_r}):=\gamma_{j-r}(\ell)$ when $j-r>1$, and take $\gamma_j(\ell|\mathbf{p_{j-1}}):=\varpi(\ell)$. Clearly, when $r\neq j-1$, the required conditions for $\gamma_j(\ell|\mathbf{p_r})$ are satisfied. When $r=j-1$, note that

$$\begin{aligned}\lambda_{\delta_1}\widetilde{\lambda}_{\widetilde{\delta}_1}\neq 0 \quad&\Rightarrow\quad p_1\cdots p_{j-1}=[\delta_1,\widetilde{\delta}_1]\leq\delta_1\widetilde{\delta}_1\leq z^{2/\varphi}\leq N^{2/\varphi}\\ &\Rightarrow\quad\frac{N}{p_1\cdots p_{j-1}}\geq N^{1-2/\varphi}\geq N^\eta>Y.\end{aligned}$$

So again the conditions are satisfied.

Lemma 4.3.2

Fix $j \in \mathbb{N}$ and let γ_j, $\gamma_j(\cdot|\mathbf{p_r})$ be defined as above. Then when we apply Theorem 2.2.2(b), all the three $\mathcal{O}$-terms that occur in the corresponding formula are $\ll \dfrac{N(\log\log N)^{\mathbb{A}}}{\log^{3k} N}$, provided that $\varphi \geq 4$. The implied constant only depends on $\mathfrak{L}$ and η.

Proof

As shown in the proof of Theorem 4.2.2, the third $\mathcal{O}$-term is

$$\ll z^{\varepsilon+2/\varphi} \ll \frac{N(\log\log N)^{\mathbb{A}}}{\log^{3k} N}.$$

The second $\mathcal{O}$-term is

$$\ll \frac{N}{Y} \sum_{q \leq N} \frac{\mu^2(q)(3k)^{\omega(q)}}{q},$$

by noting that

(i) When $\lambda_{\delta_1}\widetilde{\lambda}_{\widetilde{\delta}_1} \neq 0$, $\Delta_1 \cdots \Delta_k \leq (\delta_1 \cdots \delta_k)(\widetilde{\delta}_1 \cdots \widetilde{\delta}_k) \leq z^{2/\varphi} < N$.

(ii) Given a square-free q, there are $k^{\omega(q)}$ ways to write $q = \Delta_1 \cdots \Delta_k$, and by Lemma 2.3.1, there are $3^{\omega(\Delta_i)}$ ways to write $\Delta_i = [\delta_i, \widetilde{\delta}_i]$. So when we impose no restriction on Δ_i, altogether there are

$$k^{\omega(q)} \cdot \prod_{i=1}^{k} 3^{\omega(\Delta_i)} = (3k)^{\omega(q)}$$

ways to express q as $\Delta_1 \cdots \Delta_k$.

Hence, by Lemma 2.3.2, this $\mathcal{O}$-term is

$$\ll \frac{N \log^{\mathbb{A}} N}{Y} \ll \frac{N(\log\log N)^{\mathbb{A}}}{\log^{3k} N}.$$

Finally, we treat the first $\mathcal{O}$-term. It is

$$\ll \sum_{r=0}^{j-1} \sum_{\substack{p_1 < \cdots < p_r \leq z^{1/\varphi} \\ p_1 \cdots p_r \leq z^{2/\varphi}}} \sum_{q \leq \frac{\sqrt{\frac{a_1 N}{p_1 \cdots p_r}}}{a_1 \log^{\mathsf{B}}\left(\frac{a_1 N}{p_1 \cdots p_r}\right)}} \mu^2(q)(3k)^{\omega(q)} \mathcal{E}^*_{\gamma_j(\cdot|\mathbf{p_r})}\left(\frac{a_1 N}{p_1 \cdots p_r}; a_1 q\right),$$

by observing that

$$\begin{aligned}
\lambda_{\delta_1}\widetilde{\lambda}_{\widetilde{\delta}_1} \neq 0 \ &\Rightarrow \ a_1(\delta_1\cdots\delta_k)^2(\widetilde{\delta}_1\cdots\widetilde{\delta}_k)^2 \leq a_1 z^{4/\varphi} \leq \frac{N}{\log^{2\mathsf{B}+2} N} \\
&\Rightarrow \ a_1(\Delta_1\cdots\Delta_k)^2 \leq \frac{N}{\log^{2\mathsf{B}+2} N} \\
&\Rightarrow \ a_1\Delta_1(\Delta_2\cdots\Delta_k)^2 \leq \frac{N}{\log^{2\mathsf{B}+2} N} \\
&\Rightarrow \ a_1\Delta_2\cdots\Delta_k \leq \frac{\sqrt{\frac{a_1 N}{\Delta_1}}}{\log^{\mathsf{B}}(\frac{a_1 N}{\Delta_1})} = \frac{\sqrt{\frac{a_1 N}{p_1\cdots p_r}}}{\log^{\mathsf{B}}(\frac{a_1 N}{p_1\cdots p_r})}.
\end{aligned}$$

Hence, by Proposition 2.5.1, this $\mathcal{O}$-term is

$$\ll \sum_{r=0}^{j-1} \sum_{p_1<\cdots<p_r\leq z^{1/\varphi}} \frac{N}{p_1\cdots p_r \log^{3k} N} \ll \frac{N(\log\log N)^{\mathbb{A}}}{\log^{3k} N}.$$

Q.E.D.

We need one more lemma for the subsequent work:

<u>Lemma 4.3.3</u>

Let $\lambda^{D}_{\delta_1,\ldots,\delta_k}$ be given by Defintion 4.2.1. Suppose $D, \widetilde{D} \geq k+1$, $1 \leq \mathcal{L} \leq \mathcal{R} \leq z^{1/\varphi}$, $\mathcal{L}, \mathcal{R} \in \mathbb{N}$. Then for $(A, p_1\cdots p_r) = 1$, $p_i > Y$, we have

$$\begin{aligned}
\sum_{\substack{\delta_2,\ldots,\delta_k \\ \widetilde{\delta}_2,\ldots,\widetilde{\delta}_k \\ (\Delta_i,\Delta_j)=1 \\ (\Delta_2\cdots\Delta_k, Ap_1\cdots p_r)=1}} \frac{\lambda_{\mathcal{L}}\widetilde{\lambda}_{\mathcal{R}}}{\phi(\Delta_2\cdots\Delta_k)} = \ & \frac{\phi(A)}{A}\frac{\mu(\mathcal{L})\mu(\mathcal{R})\varphi^{k-1}\mathfrak{S}}{(\log z)^{D+\widetilde{D}-2}}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!}\Bigg[\sum_{i=0}^{D-k} \mathrm{C}_i^{D+\widetilde{D}-k-1} \\
&\times \left(\log\frac{z}{\mathcal{L}^{\varphi}}\right)^i \left(\log\frac{z}{\mathcal{R}^{\varphi}}\right)^{D+\widetilde{D}-k-1-i} \mathrm{C}_{D-k-i}^{D-2-i}(-1)^{D-k-i}\Bigg] \\
&+ \mathcal{O}\left(\frac{1}{Y\log^{k-1} N}\right) + \mathcal{O}\left(\frac{1}{\log^{k-0.5} N}\right),
\end{aligned}$$

where the implied constant only depends on $\mathfrak{L}$, φ, D, $\widetilde{D}$ and r.

Proof

$$\sum_{\substack{\delta_2,\dots,\delta_k \\ \widetilde{\delta}_2,\dots,\widetilde{\delta}_k \\ (\Delta_i,\Delta_j)=1 \\ (\Delta_2\cdots\Delta_k, Ap_1\cdots p_r)=1}} \frac{\lambda_{\mathcal{L}}\widetilde{\lambda}_{\mathcal{R}}}{\phi(\Delta_2\cdots\Delta_k)}$$

$$= \frac{\mu(\mathcal{L})\mu(\mathcal{R})}{(\log z)^{D+\widetilde{D}-2}} \sum_{\substack{\delta_2,\dots,\delta_k \\ \widetilde{\delta}_2,\dots,\widetilde{\delta}_k \\ (\Delta_i,\Delta_j)=1 \\ (\Delta_2\cdots\Delta_k, Ap_1\cdots p_r)=1}} \frac{[\mu(\delta_2)\cdots\mu(\delta_k)][\mu(\widetilde{\delta}_2)\cdots\mu(\widetilde{\delta}_k)]}{\phi(\Delta_2\cdots\Delta_k)} \left(\log\frac{\frac{z}{\mathcal{L}^\varphi}}{\delta_2^\varphi\cdots\delta_k^\varphi}\right)^{D-1} \left(\log\frac{\frac{z}{\mathcal{R}^\varphi}}{\widetilde{\delta}_2^\varphi\cdots\widetilde{\delta}_k^\varphi}\right)^{\widetilde{D}-1}$$

$$= \frac{\mu(\mathcal{L})\mu(\mathcal{R})(D-1)!(\widetilde{D}-1)!}{(\log z)^{D+\widetilde{D}-2}} \frac{1}{(2\pi i)^2} \int_{(1)}\int_{(1)} F_1 \frac{\left(\frac{z}{\mathcal{L}^\varphi}\right)^s}{s^D} \frac{\left(\frac{z}{\mathcal{R}^\varphi}\right)^{\widetilde{s}}}{\widetilde{s}^{\widetilde{D}}} ds d\widetilde{s},$$

where

$$F_1 := \sum_{\substack{\delta_2,\dots,\delta_k \\ \widetilde{\delta}_2,\dots,\widetilde{\delta}_k \\ (\Delta_i,\Delta_j)=1 \\ (\Delta_2\cdots\Delta_k, Ap_1\cdots p_r)=1}}^{\infty} \frac{[\mu(\delta_2)\cdots\mu(\delta_k)][\mu(\widetilde{\delta}_2)\cdots\mu(\widetilde{\delta}_k)]}{\phi(\Delta_2\cdots\Delta_k)} \frac{1}{(\delta_2^\varphi\cdots\delta_k^\varphi)^s} \frac{1}{(\widetilde{\delta}_2^\varphi\cdots\widetilde{\delta}_k^\varphi)^{\widetilde{s}}}$$

$$= \prod_{p\nmid Ap_1\cdots p_r} \left[1 - \frac{1}{p-1}\left(\frac{k-1}{p^{\varphi s}} + \frac{k-1}{p^{\varphi\widetilde{s}}} - \frac{k-1}{p^{\varphi s+\varphi\widetilde{s}}}\right)\right].$$

Note that we have used Lemma 2.4.4, as $z^{1/\varphi} \notin \mathbb{N}$ and

$$\sum_{\substack{\delta_2,\dots,\delta_k \\ \widetilde{\delta}_2,\dots,\widetilde{\delta}_k \\ (\Delta_i,\Delta_j)=1 \\ (\Delta_1\cdots\Delta_k, Ap_1\cdots p_r)=1}}^{\infty} \frac{|\mu(\delta_2)\cdots\mu(\delta_k)\mu(\widetilde{\delta}_2)\cdots\mu(\widetilde{\delta}_k)|}{\phi(\Delta_2\cdots\Delta_k)\delta_2^\varphi\cdots\delta_k^\varphi\widetilde{\delta}_2^\varphi\cdots\widetilde{\delta}_k^\varphi}$$

$$= \prod_{p\nmid Ap_1\cdots p_r} \left[1 + \frac{1}{p-1}\left(\frac{k-1}{p^{\varphi}} + \frac{k-1}{p^{\varphi}} + \frac{k-1}{p^{2\varphi}}\right)\right] < \infty.$$

Let

$$g_1(p) := \left(1 - \frac{1}{p^{1+\varphi s}}\right)^{-(k-1)} \left(1 - \frac{1}{p^{1+\varphi\widetilde{s}}}\right)^{-(k-1)} \left(1 - \frac{1}{p^{1+\varphi s+\varphi\widetilde{s}}}\right)^{k-1},$$

so that

$$F_1 = \left\{F_1 \cdot \prod_p g_1(p)\right\} \prod_p g(p)^{-1}$$

$$:= \{G_1\} \frac{\zeta^{k-1}(1+\varphi s+\varphi\widetilde{s})}{\zeta^{k-1}(1+\varphi s)\zeta^{k-1}(1+\varphi\widetilde{s})}.$$

By Lemma 2.3.5, Proposition 2.4.2, and Lemma 4.1.1 with $\kappa = k-1$,

$$\begin{aligned}
&\frac{1}{(2\pi i)^2}\int_{(1)}\int_{(1)} F_1 \frac{\left(\frac{z}{\mathcal{L}\varphi}\right)^s}{s^D}\frac{\left(\frac{z}{\mathcal{R}\varphi}\right)^{\widetilde{s}}}{\widetilde{s}^{\widetilde{D}}} \\
= \;& \frac{\varphi^{k-1}G_1(\mathbf{0})}{(D+\widetilde{D}-k-1)!}\sum_{i=0}^{D-k} \mathrm{C}_i^{D+\widetilde{D}-k-1}\left(\log\frac{z}{\mathcal{L}\varphi}\right)^i\left(\log\frac{z}{\mathcal{R}\varphi}\right)^{D+\widetilde{D}-k-1-i}\mathrm{C}_{D-k-i}^{D-2-i}(-1)^{D-k-i} \\
&+\mathcal{O}\Big((\log N)^{D+\widetilde{D}-k-1.5}\Big).
\end{aligned}$$

Since

$$\begin{aligned}
G_1(\mathbf{0}) &= \prod_{p\nmid Ap_1\cdots p_r}\left(1-\frac{k-1}{p-1}\right)\left(1-\frac{1}{p}\right)^{-(k-1)}\prod_{p\mid Ap_1\cdots p_r}\left(1-\frac{1}{p}\right)^{-(k-1)} \\
&= \prod_{p\nmid A}\frac{p^{k-1}(p-k)}{(p-1)^k}\prod_{p\mid p_1\cdots p_r}\frac{(p-1)^k}{p^{k-1}(p-k)}\prod_{p\mid Ap_1\cdots p_r}\left(1-\frac{1}{p}\right)^{-(k-1)} \\
&= \prod_{p\nmid A}\left(1-\frac{1}{p}\right)^{-k}\left(1-\frac{k}{p}\right)\prod_{p\mid p_1\cdots p_r}\frac{p-1}{p-k}\prod_{p\mid A}\left(1-\frac{1}{p}\right)^{-(k-1)} \\
&= \left[\prod_{p\nmid A}\left(1-\frac{1}{p}\right)^{-k}\left(1-\frac{k}{p}\right)\prod_{p\mid A}\left(1-\frac{1}{p}\right)^{-k}\right]\prod_{p\mid A}\left(1-\frac{1}{p}\right)\prod_{p\mid p_1\cdots p_r}\frac{p-1}{p-k} \\
&= \mathfrak{S}\frac{\phi(A)}{A}\prod_{p\mid p_1\cdots p_r}\left(1+\mathcal{O}\left(\frac{1}{Y}\right)\right),
\end{aligned}$$

the result follows.

Q.E.D.

4.4 Asymptotic formulas for Q_1 and Q_2

<u>**Definition 4.4.1**</u>

We write "$\mathcal{L} \doteqdot \mathcal{R}$" if

$$\mathcal{L} = \mathcal{R} + \mathcal{O}\left(\frac{N(\log\log N)^{\mathbb{A}}}{(\log N)^{k+0.5}}\right),$$

where the implied constant only depends on $\mathfrak{L}$, D, $\widetilde{D}$, φ, j, η.

Theorem 4.4.2

Let $\mathfrak{L}$ be HB-admissible, and let

$$Q_1 := \sum_{N<n\leq 2N} \gamma_1(a_1n+b_1) \left(\sum_{\delta_i | a_in+b_i} \lambda^D_{\delta_1,\ldots,\delta_k} \right) \left(\sum_{\widetilde{\delta}_i | a_in+b_i} \lambda^{\widetilde{D}}_{\widetilde{\delta}_1,\ldots,\widetilde{\delta}_k} \right),$$

where γ_1 is given by Definition 4.3.1 and $\lambda^D_{\delta_1,\ldots,\delta_k}$ is given by Defintion 4.2.1. Suppose D, $\widetilde{D} \geq k+1$, and $\varphi \geq 4$. Then

$$Q_1 \doteqdot \frac{\varphi^{k-1}\mathfrak{S}N}{\log^k N} \frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!} \mathrm{C}^{D+\widetilde{D}-2k}_{D-k}.$$

Proof

We apply Theorem 2.2.2(b) and Lemma 4.3.2 with $j=1$. It follows that

$$Q_1 \doteqdot \sum_{a_1N<\ell\leq 2a_1N} \varpi(\ell) \sum_{\substack{\delta_2,\ldots,\delta_k \\ \widetilde{\delta}_2,\ldots,\widetilde{\delta}_k \\ (\Delta_i,\Delta_j)=1 \\ (\Delta_2\cdots\Delta_k,A\ell)=1}} \frac{\lambda_1\widetilde{\lambda}_1}{\phi(a_1\Delta_2\cdots\Delta_k)}.$$

For $\varpi(\ell)\neq 0$, by Lemma 4.3.3 ($r=1$) together with (4.4),

$$\sum_{\substack{\delta_2,\ldots,\delta_k \\ \widetilde{\delta}_2,\ldots,\widetilde{\delta}_k \\ (\Delta_i,\Delta_j)=1 \\ (\Delta_2\cdots\Delta_k,A\ell)=1}} \frac{\lambda_1\widetilde{\lambda}_1}{\phi(\Delta_2\cdots\Delta_k)} = \frac{\phi(A)}{A}\frac{\varphi^{k-1}\mathfrak{S}}{\log^{k-1}z}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!}\mathrm{C}^{D+\widetilde{D}-2k}_{D-k} + \mathcal{O}\left(\frac{1}{Y\log^{k-1}N}\right) + \mathcal{O}\left(\frac{1}{\log^{k-0.5}N}\right).$$

This gives

$$\begin{aligned} Q_1 &\doteqdot \sum_{a_1N<\ell\leq 2a_1N} \frac{\varpi(\ell)}{\phi(a_1)}\frac{\phi(A)}{A}\frac{\varphi^{k-1}\mathfrak{S}}{\log^{k-1}z}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!}\mathrm{C}^{D+\widetilde{D}-2k}_{D-k} \\ &\quad + \mathcal{O}\left(\sum_{a_1N<\ell\leq 2a_1N}\frac{\varpi(\ell)}{\log^{k-0.5}N}\right). \end{aligned}$$

We then apply the prime number theorem. Note that since $\mathfrak{L}$ is HB-admissible, we have $\dfrac{a_1}{\phi(a_1)}\dfrac{\phi(A)}{A}=1$. It follows that

$$\begin{aligned} Q_1 &\doteqdot \frac{\varphi^{k-1}\mathfrak{S}N}{\log N\log^{k-1}z}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!}\mathrm{C}^{D+\widetilde{D}-2k}_{D-k} + \mathcal{O}\left(\frac{N}{\log^{k+0.5}N}\right) \\ &\doteqdot \frac{\varphi^{k-1}\mathfrak{S}N}{\log^k N}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!}\mathrm{C}^{D+\widetilde{D}-2k}_{D-k}. \end{aligned}$$

Q.E.D.

Theorem 4.4.3

Let $\mathfrak{L}$ be HB-admissible, and let

$$Q_2 := \sum_{N<n\le 2N} \gamma_2(a_1n+b_1)\left(\sum_{\delta_i|a_in+b_i} \lambda^{D}_{\delta_1,\ldots,\delta_k}\right)\left(\sum_{\widetilde{\delta}_i|a_in+b_i} \lambda^{\widetilde{D}}_{\widetilde{\delta}_1,\ldots,\widetilde{\delta}_k}\right),$$

where γ_2 is given by Definition 4.3.1 and $\lambda^{D}_{\delta_1,\ldots,\delta_k}$ is given by Defintion 4.2.1. Suppose D, $\widetilde{D}\ge k+1$, and $\varphi\ge 4$. Then

$$\begin{aligned} Q_2 \;\doteqdot\; & \frac{\varphi^k\mathfrak{S}N}{(\log N)^k}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!}\Bigg\{\mathrm{C}^{D+\widetilde{D}-2k}_{D-k}\int_{\eta}^{\frac{\varphi}{2}}\frac{1}{x_1(\varphi-x_1)}dx_1 \\ & -\int_{\eta}^{1}\frac{1}{x_1(\varphi-x_1)}\left[\sum_{i=0}^{D-k}\mathrm{C}^{D+\widetilde{D}-k-1}_{i}\mathrm{C}^{D-2-i}_{D-k-i}(-1)^{D-k-i}(1-x_1)^{D+\widetilde{D}-k-1-i}\right]dx_1 \\ & -\int_{\eta}^{1}\frac{1}{x_1(\varphi-x_1)}\left[\sum_{i=0}^{\widetilde{D}-k}\mathrm{C}^{D+\widetilde{D}-k-1}_{i}\mathrm{C}^{\widetilde{D}-2-i}_{\widetilde{D}-k-i}(-1)^{\widetilde{D}-k-i}(1-x_1)^{D+\widetilde{D}-k-1-i}\right]dx_1 \\ & +\mathrm{C}^{D+\widetilde{D}-2k}_{D-k}\int_{\eta}^{1}\frac{1}{x_1(\varphi-x_1)}(1-x_1)^{D+\widetilde{D}-k-1}dx_1\Bigg\}. \end{aligned}$$

Proof

We apply Theorem 2.2.2(b) and Lemma 4.3.2 with $j=2$. It follows that

$$\begin{aligned} Q_2 \;\doteqdot\; & \sum_{a_1N<\ell\le 2a_1N}\gamma_2(\ell)\sum_{\substack{\delta_2,\ldots,\delta_k\\ \widetilde{\delta}_2,\ldots,\widetilde{\delta}_k\\ (\Delta_i,\Delta_j)=1\\ (\Delta_2\cdots\Delta_k,A\ell)=1}}\frac{[\lambda_1\widetilde{\lambda}_1]}{\phi(a_1\Delta_2\cdots\Delta_k)} \\ & +\sum_{Y<p_1\le z^{1/\varphi}}\sum_{\frac{a_1N}{p_1}<\ell\le\frac{2a_1N}{p_1}}\varpi(\ell)\sum_{\substack{\delta_2,\ldots,\delta_k\\ \widetilde{\delta}_2,\ldots,\widetilde{\delta}_k\\ (\Delta_i,\Delta_j)=1\\ (\Delta_2\cdots\Delta_k,Ap_1\ell)=1}}\frac{[\lambda_1\widetilde{\lambda}_{p_1}+\lambda_{p_1}\widetilde{\lambda}_1+\lambda_{p_1}\widetilde{\lambda}_{p_1}]}{\phi(a_1\Delta_2\cdots\Delta_k)}. \end{aligned}$$

We now use Lemma 4.3.3 and (4.4). This gives

$$
\begin{aligned}
Q_2 \ \doteqdot & \sum_{a_1N<\ell\le 2a_1N} \frac{\gamma_2(\ell)}{\phi(a_1)}\frac{\phi(A)}{A}\frac{\varphi^{k-1}\mathfrak{S}}{(\log z)^{k-1}}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!}\mathrm{C}_{D-k}^{D+\widetilde{D}-2k} \\
& + \sum_{Y<p_1\le z^{1/\varphi}} \sum_{\frac{a_1N}{p_1}<\ell\le\frac{2a_1N}{p_1}} \frac{\varpi(\ell)}{\phi(a_1)}\frac{\phi(A)}{A}\frac{\varphi^{k-1}\mathfrak{S}}{(\log z)^{D+\widetilde{D}-2}}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!} \times \\
& \quad \Bigg\{ -\sum_{i=0}^{D-k}\mathrm{C}_i^{D+\widetilde{D}-k-1}(\log z)^i\left(\log\frac{z}{p_1^{\varphi}}\right)^{D+\widetilde{D}-k-1-i}\mathrm{C}_{D-k-i}^{D-2-i}(-1)^{D-k-i} \\
& \quad -\sum_{i=0}^{\widetilde{D}-k}\mathrm{C}_i^{D+\widetilde{D}-k-1}(\log z)^i\left(\log\frac{z}{p_1^{\varphi}}\right)^{D+\widetilde{D}-k-1-i}\mathrm{C}_{\widetilde{D}-k-i}^{\widetilde{D}-2-i}(-1)^{\widetilde{D}-k-i} \\
& \quad +\mathrm{C}_{D-k}^{D-\widetilde{D}-2k}\left(\log\frac{z}{p_1^{\varphi}}\right)^{D+\widetilde{D}-k-1}\Bigg\} \\
& + \mathcal{O}\left(\sum_{a_1N<\ell\le 2a_1N}\frac{\gamma_2(\ell)}{\log^{k-0.5}N}\right) + \mathcal{O}\left(\sum_{Y<p_1\le z^{1/\varphi}}\sum_{\frac{a_1N}{p_1}<\ell\le\frac{2a_1N}{p_1}}\frac{\varpi(\ell)}{\log^{k-0.5}N}\right).
\end{aligned}
$$

By the prime number theorem, the last $\mathcal{O}$-term is $\ll \dfrac{N\log\log N}{(\log N)^{k+0.5}}$, which can be absorbed in $\doteqdot$. So is the first $\mathcal{O}$-term, because

$$
\sum_{a_1N<\ell\le 2a_1N}\gamma_2(\ell) \le \sum_{Y<p_1\le N^{\varepsilon+1/2}}\sum_{\frac{a_1N}{p_1}<\ell\le\frac{2a_1N}{p_1}}\varpi(\ell) \ll \frac{N\log\log N}{\log N}.
$$

It remains to treat the first two terms. Observe that

$$
\begin{aligned}
\sum_{a_1N<\ell\le 2a_1N}\gamma_2(\ell) &= \sum_{Y<p_1\le N^{1/2}}\sum_{\frac{a_1N}{p_1}<p_2\le\frac{2a_1N}{p_1}}1 + \sum_{N^{1/2}<p_1}\sum_{\substack{\frac{a_1N}{p_1}<p_2\le\frac{2a_1N}{p_1}\\ p_1<p_2}}1 \\
&= \sum_{Y<p_1\le N^{1/2}}\frac{a_1N}{p_1\log(N/p_1)} + \mathcal{O}\left(\frac{N\log\log N}{\log^2 N}\right),
\end{aligned}
$$

because the last double sum is

$$
\ll \sum_{N^{\frac12}<p_1<\sqrt{2a_1N}}\ \sum_{p_1<p_2\le\frac{2a_1N}{p_1}}1 \ll \sum_{N^{\frac12}<p_1<\sqrt{2a_1N}}\ \sum_{N^{\frac12}<p_2\le 2a_1N^{\frac12}}1 \ll \frac{N}{\log^2 N}.
$$

Therefore,

$$\begin{aligned}
Q_2 \doteqdot\ & \frac{\varphi^{k-1}\mathfrak{S}N}{\log^{k-1}z}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!}\mathrm{C}_{D-k}^{D+\widetilde{D}-2k}\sum_{Y<p_1\le N^{1/2}}\frac{1}{p_1\log(N/p_1)}\\
&+\frac{\varphi^{k-1}\mathfrak{S}N}{\log^{D+\widetilde{D}-2}z}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!}\sum_{Y<p_1\le z^{1/\varphi}}\frac{1}{p_1\log(N/p_1)}\Bigg\{\\
&\qquad -\sum_{i=0}^{D-k}\mathrm{C}_i^{D+\widetilde{D}-k-1}(\log z)^i\left(\log\frac{z}{p_1^{\varphi}}\right)^{D+\widetilde{D}-k-1-i}\mathrm{C}_{D-k-i}^{D-2-i}(-1)^{D-k-i}\\
&\qquad -[D\rightleftarrows\widetilde{D}]\ +\ \mathrm{C}_{D-k}^{D+\widetilde{D}-2k}\left(\log\frac{z}{p_1^{\varphi}}\right)^{D+\widetilde{D}-k-1}\Bigg\},
\end{aligned}$$

where the meaning of $[D \rightleftarrows \widetilde{D}]$ should be clear. Here we note that when this symbol occurs later, its meaning may differ but should also be clear from the context.

We now expand $(\log z-\varphi\log p_1)^m$ in the braces by binomial theorem. Each term in the sums over p_1 is then of the form

$$a(\log z)^n V_1(p_1),$$

where

$$V_1(y_1)=\frac{(\log y_1)^{m_1}}{y_1\log(N/y_1)}.$$

We go to apply Proposition 2.4.8. By Proposition 2.4.9 the contributions from R_i's altogether are $\ll \dfrac{N}{\log^{\mathbb{A}}N}$, which is acceptable. We then group the resulting V_0's by reversing the binomial expansions. This results in

$$\begin{aligned}
Q_2 \doteqdot\ & \frac{\varphi^{k-1}\mathfrak{S}N}{\log^{k-1}z}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!}\mathrm{C}_{D-k}^{D+\widetilde{D}-2k}\int_Y^{N^{\frac{1}{2}}}\frac{1}{(u_1\log u_1)\log(N/u_1)}du_1\\
&+\frac{\varphi^{k-1}\mathfrak{S}N}{\log^{D+\widetilde{D}-2}z}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!}\int_Y^{z^{1/\varphi}}\frac{1}{(u_1\log u_1)\log(N/u_1)}\Bigg\{\\
&\qquad -\sum_{i=0}^{D-k}\mathrm{C}_i^{D+\widetilde{D}-k-1}(\log z)^i\left(\log\frac{z}{u_1^{\varphi}}\right)^{D+\widetilde{D}-k-1-i}\mathrm{C}_{D-k-i}^{D-2-i}(-1)^{D-k-i}\\
&\qquad -[D\rightleftarrows\widetilde{D}]\ +\ \mathrm{C}_{D-k}^{D+\widetilde{D}-2k}\left(\log\frac{z}{u_1^{\varphi}}\right)^{D+\widetilde{D}-k-1}\Bigg\}du_1.
\end{aligned}$$

We next use the change of variable $u_1=z^{x_1/\varphi}$. Noting that

$$\frac{du_1}{u_1\log u_1}=\frac{(z^{x_1/\varphi})\frac{\log z}{\varphi}dx_1}{(z^{x_1/\varphi})\frac{x_1}{\varphi}\log z}=\frac{dx_1}{x_1},$$

we have

$$
\begin{aligned}
Q_2 \;\doteqdot\; & \frac{\varphi^{k-1}\mathfrak{S}N}{\log^{k-1}z}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!}\mathrm{C}_{D-k}^{D+\widetilde{D}-2k}\int\limits_{\eta}^{\frac{\varphi\log N}{2\log z}}\frac{\varphi}{x_1(\varphi\log N-x_1\log z)}dx_1 \\
& + \frac{\varphi^{k-1}\mathfrak{S}N}{\log^{k-1}z}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!}\int\limits_{\eta}^{1}\frac{\varphi}{x_1(\varphi\log N-x_1\log z)}\Bigg\{ \\
& \qquad -\sum_{i=0}^{D-k}\mathrm{C}_i^{D+\widetilde{D}-k-1}\mathrm{C}_{D-k-i}^{D-2-i}(-1)^{D-k-i}(1-x_1)^{D+\widetilde{D}-k-1-i} \\
& \qquad -[D\rightleftarrows\widetilde{D}] \;+\; \mathrm{C}_{D-k}^{D+\widetilde{D}-2k}(1-x_1)^{D+\widetilde{D}-k-1}\Bigg\}dx_1.
\end{aligned}
$$

By using

$$
\begin{aligned}
\frac{1}{\varphi\log N-x_1\log z} &= \frac{1}{\varphi\log N-x_1\log N+\mathsf{C}x_1\log\log N} \\
&= \frac{1}{(\varphi-x_1)\log N}\left(1+\mathcal{O}\left(\frac{\log\log N}{\log N}\right)\right)
\end{aligned}
$$

uniformly in the range of x_1 under consideration, and then using

$$
\frac{\varphi\log N}{2\log z}=\frac{\varphi}{2}\left(1+\mathcal{O}\left(\frac{\log\log N}{\log N}\right)\right),
$$

we see that

$$
\begin{aligned}
Q_2 \;\doteqdot\; & \frac{\varphi^{k}\mathfrak{S}N}{\log N\log^{k-1}z}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!}\mathrm{C}_{D-k}^{D+\widetilde{D}-2k}\int\limits_{\eta}^{\frac{\varphi}{2}}\frac{1}{x_1(\varphi-x_1)}dx_1 \\
& + \frac{\varphi^{k}\mathfrak{S}N}{\log N\log^{k-1}z}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!}\int\limits_{\eta}^{1}\frac{1}{x_1(\varphi-x_1)}\Bigg\{ \\
& \qquad -\sum_{i=0}^{D-k}\mathrm{C}_i^{D+\widetilde{D}-k-1}\mathrm{C}_{D-k-i}^{D-2-i}(-1)^{D-k-i}(1-x_1)^{D+\widetilde{D}-k-1-i} \\
& \qquad -[D\rightleftarrows\widetilde{D}] \;+\; \mathrm{C}_{D-k}^{D+\widetilde{D}-2k}(1-x_1)^{D+\widetilde{D}-k-1}\Bigg\}dx_1.
\end{aligned}
$$

The result follows.

Q.E.D.

4.5 Interlude: gaps between prime numbers

In this section, we shall discuss in a relaxed manner the existence of narrow gaps between prime numbers.

To begin, recall that we have come up with the following asymptotic formulas:

$$\sum_{N<n\le 2N}\left(\sum_{\delta_i|a_in+b_i}\lambda^D_{\delta_1,\dots,\delta_k}\right)^2 \sim \frac{\varphi^k\mathfrak{S}N}{\log^k N}\frac{[(D-1)!]^2}{(2D-k-2)!}\mathrm{C}^{2D-2k-2}_{D-k-1},$$

$$\sum_{N<n\le 2N}\varpi(a_in+b_i)\left(\sum_{\delta_i|a_in+b_i}\lambda^D_{\delta_1,\dots,\delta_k}\right)^2 \sim \frac{\varphi^{k-1}\mathfrak{S}N}{\log^k N}\frac{[(D-1)!]^2}{(2D-k-1)!}\mathrm{C}^{2D-2k}_{D-k},$$

which are the results of Theorem 4.2.2 and Theorem 4.4.2.

Let's assume for the time being that $a_1 = a_2 = \cdots = a_k$. Now, if one can show that for some k,

$$\sum_{N<n\le 2N}[\varpi(a_1n+b_1)+\cdots+\varpi(a_kn+b_k)-1]\left(\sum_{\delta_i|a_in+b_i}\lambda^D_{\delta_1,\dots,\delta_k}\right)^2 > 0$$

for all arbitrarily large N, then we would obtain an important result in number theory, namely

$$\liminf_{n\to\infty}(p_{n+1}-p_n) \le \max\{|b_i-b_j|\} < \infty,$$

where p_n denotes the n^{th} prime.

To achieve this, we try to show

$$\frac{k}{\varphi}\frac{\mathrm{C}^{2D-2k}_{D-k}}{(2D-k-1)} - \mathrm{C}^{2D-2k-2}_{D-k-1} > 0$$

for some k and $D \ge k+1$, as the asymptotic formulas suggest. Note that the left hand side is

$$= \mathrm{C}^{2D-2k-2}_{D-k-1}\left[\frac{k}{\varphi}\frac{(2D-2k)(2D-2k-1)}{(D-k)(D-k)(2D-k-1)} - 1\right].$$

When $D \sim k+\sqrt{k}$, k sufficiently large, the terms in the square brackets are

$$\sim \frac{k}{\varphi}\frac{2\sqrt{k}\cdot 2\sqrt{k}}{\sqrt{k}\cdot\sqrt{k}\cdot k} - 1 = \frac{4}{\varphi} - 1.$$

This demonstrates that, if we are allowed to take $\varphi < 4$, then we would succeed in showing that bounded gaps between consecutive primes occur infinitely many often.

Our established results, however, only allow that φ is not less than 4. The reason is as follows. Observe that φ corresponds to the truncation of our Λ^2 sieve (see Defintion 4.2.1). When evaluating the asymptotic formula for Q_1, on the one hand we have used Bombieri-Vinogradov Theorem, which allows the level of distribution to be $\frac{1}{2}$; while on the other hand, we sacrifice half such information (due to the presence of the square in Λ^2) to ensure a nonnegative sieve. Therefore, the truncation parameter φ is required to be at least 4.

Nevertheless, as we have just seen, we are close to proving the statement $\liminf_{n\to\infty}(p_{n+1}-p_n) < \infty$, and this can be achieved if one makes a hypothesis that the level of distribution can be $\geq \frac{1}{2}+\varepsilon$.

To compensate the deficiency of an ε unconditionally, perhaps one may think of the expression

$$\sum_{N<n\leq 2N}\left[\sum_{b<h}\varpi(a_1n+b)-1\right]\left(\sum_{\delta_i|a_in+b_i}\lambda^D_{\delta_1,\ldots,\delta_k}\right)^2$$

with $h=\varepsilon\log N$, as suggested by the prime number theorem. Observe that this corresponds to a result of the type

$$\liminf_{n\to\infty}\frac{p_{n+1}-p_n}{\log p_n}\leq\varepsilon.$$

Indeed, by considering a similar sum, and also summing over different admissible k-tuples for an averaging effect, the result in [7] is that

$$\liminf_{n\to\infty}\frac{p_{n+1}-p_n}{\log p_n}=0,$$

which can be interpreted as follows [17] :"there are infinitely many primes for which the gap to the next prime is as small as we want compared to the average gap between consecutive primes". As a result, we know that small gaps between primes exist.

How about bounded gaps? In his breakthrough paper[21], Zhang succeeds in overcoming the threshold $\frac{1}{2}$ for the Bombieri-Vinogradov theorem when the moduli are only composed by small primes. We need to impose a constraint on the sieve in order that this Bombieri-Vinogradov type theorem can be used, and it turns out that the gain in the level of distribution outweighs the loss due to the constraint. As a result, we have a more powerful sieve, and this leads to the answer that bounded gaps between primes exist.

4.6 Towards $j \geq 3$

We now consider a general $j \geq 3$, and fix $\varphi = 4$ from now on.

By Theorem 2.2.2(b) and Lemma 4.3.2,

$$Q_j \doteqdot \sum_{r=0}^{j-1} \sum_{Y<p_1<\cdots<p_r\leq z^{\frac{1}{4}}} \sum_{\frac{a_1N}{p_1\cdots p_r}<\ell\leq\frac{2a_1N}{p_1\cdots p_r}} \gamma_j(\ell|\mathbf{p_r}) \sum_{\substack{\delta_2,\ldots,\delta_k \\ \widetilde{\delta}_2,\ldots,\widetilde{\delta}_k \\ (\Delta_i,\Delta_j)=1 \\ (\Delta_2\cdots\Delta_k, Ap_1\cdots p_r\ell)=1}} \frac{\left[\sum_{[\delta_1,\widetilde{\delta}_1]=p_1\cdots p_r} \lambda_{\delta_1}\widetilde{\lambda}_{\widetilde{\delta}_1}\right]}{\phi(a_1\Delta_2\cdots\Delta_k)}.$$

By Lemma 4.3.3, the right hand side is

$$\begin{aligned}
&\doteqdot \sum_{r=0}^{j-1} \sum_{Y<p_1<\cdots<p_r\leq z^{\frac{1}{4}}} \sum_{\frac{a_1N}{p_1\cdots p_r}<\ell\leq\frac{2a_1N}{p_1\cdots p_r}} \frac{\gamma_j(\ell|\mathbf{p_r})}{\phi(a_1)} \frac{\phi(A)}{A} \frac{4^{k-1}\mathfrak{S}}{(\log z)^{D+\widetilde{D}-2}} \frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!} \times \\
&\left\{ \sum_{\substack{[\delta_1,\widetilde{\delta}_1]=p_1\cdots p_r \\ \delta_1\leq\widetilde{\delta}_1\leq z^{\frac{1}{4}}}} \mu(\delta_1)\mu(\widetilde{\delta}_1) \sum_{i=0}^{D-k} \mathrm{C}_i^{D+\widetilde{D}-k-1} \left(\log\frac{z}{\delta_1^4}\right)^i \left(\log\frac{z}{\widetilde{\delta}_1^4}\right)^{D+\widetilde{D}-k-1-i} \mathrm{C}_{D-k-i}^{D-2-i}(-1)^{D-k-i} \right. \\
&\left. + \sum_{\substack{[\delta_1,\widetilde{\delta}_1]=p_1\cdots p_r \\ \widetilde{\delta}_1<\delta_1\leq z^{\frac{1}{4}}}} \mu(\delta_1)\mu(\widetilde{\delta}_1) \sum_{i=0}^{\widetilde{D}-k} \mathrm{C}_i^{D+\widetilde{D}-k-1} \left(\log\frac{z}{\widetilde{\delta}_1^4}\right)^i \left(\log\frac{z}{\delta_1^4}\right)^{D+\widetilde{D}-k-1-i} \mathrm{C}_{\widetilde{D}-k-i}^{\widetilde{D}-2-i}(-1)^{\widetilde{D}-k-i} \right\} \\
&+ \mathcal{O}\left(\sum_{r=0}^{j-1} \sum_{\substack{Y<p_1<\cdots<p_r\leq z^{\frac{1}{4}} \\ p_1\cdots p_r\leq z^{2/4}}} \sum_{\frac{a_1N}{p_1\cdots p_r}<\ell\leq\frac{2a_1N}{p_1\cdots p_r}} \frac{\gamma_j(\ell|\mathbf{p_r})}{(\log N)^{k-0.5}} \right),
\end{aligned}$$

where the condition $p_1\cdots p_r \leq z^{2/4}$ in the $\mathcal{O}$-term is due to the requirement $\lambda_{\delta_1}\widetilde{\lambda}_{\widetilde{\delta}_1} \neq 0$ for $[\delta_1, \widetilde{\delta}_1] = p_1\cdots p_r$.

We shall write $[D \rightleftarrows \widetilde{D}]$ for the second sum in the braces, because by symmetry it suffices to focus on the first sum inside. Similar to before, we shall use $[D \rightleftarrows \widetilde{D}]$ generally, so that its meaning varies each time.

By letting

$$S_r = S_r(j) := \sum_{\frac{a_1N}{p_1\cdots p_r}<\ell\leq\frac{2a_1N}{p_1\cdots p_r}} \gamma_j(\ell|\mathbf{p_r}),$$

we get

(4.5)

$$Q_j \doteqdot \sum_{r=0}^{j-1} \sum_{Y<p_1<\cdots<p_r\leq z^{\frac{1}{4}}} \frac{S_r}{\phi(a_1)} \frac{\phi(A)}{A} \frac{4^{k-1}\mathfrak{S}}{(\log z)^{D+\widetilde{D}-2}} \frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!} \times$$

$$\left\{ \sum_{\substack{[\delta_1,\widetilde{\delta}_1]=p_1\cdots p_r \\ \delta_1\leq\widetilde{\delta}_1\leq z^{\frac{1}{4}}}} \mu(\delta_1)\mu(\widetilde{\delta}_1) \sum_{i=0}^{D-k} \mathrm{C}_i^{D+\widetilde{D}-k-1} \left(\log\frac{z}{\delta_1^4}\right)^i \left(\log\frac{z}{\widetilde{\delta}_1^4}\right)^{D+\widetilde{D}-k-1-i} \mathrm{C}_{D-k-i}^{D-2-i}(-1)^{D-k-i} \right\}$$

$$+ [D \rightleftarrows \widetilde{D}] + \mathcal{O}\left(\sum_{r=0}^{j-1} \sum_{\substack{Y<p_1<\cdots<p_r\leq z^{\frac{1}{4}} \\ p_1\cdots p_r\leq z^{1/2}}} \frac{S_r}{(\log N)^{k-0.5}} \right).$$

When $0 \leq r \leq j-2$, we split S_r into

$$\sum_{\substack{\frac{a_1 N}{p_1\cdots p_r}<\ell\leq\frac{2a_1 N}{p_1\cdots p_r} \\ \ell=p_{r+1}\cdots p_j,\, p_{r+1}<\cdots<p_j \\ p_1\cdots p_{j-1}^2\leq N}} \gamma_j(\ell|\mathbf{p_r}) + \sum_{\substack{\frac{a_1 N}{p_1\cdots p_r}<\ell\leq\frac{2a_1 N}{p_1\cdots p_r} \\ \ell=p_{r+1}\cdots p_j,\, p_{r+1}<\cdots<p_j \\ N<p_1\cdots p_{j-1}^2}} \gamma_j(\ell|\mathbf{p_r}) := S_{r,M} + S_{r,\mathcal{O}}.$$

We go to show that the contribution from $S_{r,\mathcal{O}}$'s can be absorbed in $\doteqdot$.

Lemma 4.6.1

$$\sum_{Y<p_1<\cdots<p_r\leq z^{\frac{1}{4}}} S_{r,\mathcal{O}} \ll \frac{N(\log\log N)^{\mathbb{A}}}{\log^2 N}.$$

Proof

Clearly,

$$\sum_{Y<p_1<\cdots<p_r\leq z^{\frac{1}{4}}} S_{r,\mathcal{O}} \leq \sum_{Y<p_1<N} \cdots \sum_{Y<p_{j-2}<N} \sum_{\substack{Y<p_{j-1}<N \\ \sqrt{\frac{N}{p_1\cdots p_{j-2}}}<p_{j-1}}} \sum_{\substack{\frac{a_1 N}{p_1\cdots p_{j-1}}<p_j\leq\frac{2a_1 N}{p_1\cdots p_{j-1}} \\ p_{j-1}<p_j}} 1. \tag{4.6}$$

Since

$$\begin{cases} p_j \leq \dfrac{2a_1 N}{p_1\cdots p_{j-1}} \\ Y < p_{j-1} < p_j \end{cases} \Rightarrow \sqrt{\frac{N}{p_1\cdots p_{j-2}}} > \frac{Y}{\sqrt{2a_1}} > Y^{\frac{1}{2}},$$

the right hand side of (4.6) is

$$\begin{aligned}
&\leq \sum_{Y<p_1<N} \cdots \sum_{\substack{Y<p_{j-2}<N \\ \sqrt{\frac{N}{p_1\cdots p_{j-2}}}>Y^{\frac{1}{2}}}} \sum_{\sqrt{\frac{N}{p_1\cdots p_{j-2}}}<p_{j-1}<N} \sum_{p_{j-1}<p_j\leq\frac{2a_1N}{p_1\cdots p_{j-1}}} 1 \\
&\leq \sum_{Y<p_1<N} \cdots \sum_{\substack{Y<p_{j-2}<N \\ \sqrt{\frac{N}{p_1\cdots p_{j-2}}}>Y^{\frac{1}{2}}}} \sum_{\sqrt{\frac{N}{p_1\cdots p_{j-2}}}<p_{j-1}<\sqrt{\frac{2a_1N}{p_1\cdots p_{j-2}}}} \sum_{\sqrt{\frac{N}{p_1\cdots p_{j-2}}}<p_j\leq 2a_1\sqrt{\frac{N}{p_1\cdots p_{j-2}}}} 1 \\
&\ll \sum_{Y<p_1<N} \cdots \sum_{\substack{Y<p_{j-2}<N \\ \sqrt{\frac{N}{p_1\cdots p_{j-2}}}>Y^{\frac{1}{2}}}} \sum_{\sqrt{\frac{N}{p_1\cdots p_{j-2}}}<p_{j-1}<\sqrt{\frac{2a_1N}{p_1\cdots p_{j-2}}}} \frac{\sqrt{\frac{N}{p_1\cdots p_{j-2}}}}{\log Y} \\
&\ll \sum_{Y<p_1<N} \cdots \sum_{\substack{Y<p_{j-2}<N \\ \sqrt{\frac{N}{p_1\cdots p_{j-2}}}>Y^{\frac{1}{2}}}} \frac{N}{p_1\cdots p_{j-2}\log^2 Y} \ll \frac{N(\log\log N)^{\mathbb{A}}}{\log^2 N}.
\end{aligned}$$

Q.E.D.

The contribution of $S_{r,\mathcal{O}}$ to Q_j in (4.5) is therefore

$$\ll \sum_{r=0}^{j-2} \sum_{Y<p_1<\cdots<p_r\leq z^{\frac{1}{4}}} \frac{(\log z)^{D+\tilde{D}-k-1}}{(\log z)^{D+\tilde{D}-2}} S_{r,\mathcal{O}} \ll \frac{N(\log\log N)^{\mathbb{A}}}{(\log N)^{k+1}},$$

which can be absorbed in $\doteqdot$. Thus, by writing $S_{j-1,M} = S_{j-1}$, we can replace every S_r occurred in (4.5) by $S_{r,M}$.

Denote by $\mathsf{C}_j(r)$ the condition $p_1\cdots p_{j-1}^2 \leq N$ if $0 \leq r \leq j-2$, and be void if $r = j-1$. The $\mathcal{O}$-term in (4.5) is

$$\begin{aligned}
&\ll \sum_{r=0}^{j-1} \sum_{\substack{Y<p_1<\cdots<p_r\leq z^{\frac{1}{4}} \\ p_1\cdots p_r\leq z^{1/2} \\ Y<p_{r+1}<\cdots<p_{j-1} \\ \mathsf{C}_j(r)}} \sum_{\frac{a_1N}{p_1\cdots p_{j-1}}<p_j\leq\frac{2a_1N}{p_1\cdots p_{j-1}}} \frac{1}{(\log N)^{k-0.5}} \\
&\ll \sum_{r=0}^{j-1} \sum_{\substack{Y<p_1<\cdots<p_r\leq z^{\frac{1}{4}} \\ p_1\cdots p_r\leq z^{1/2} \\ Y<p_{r+1}<\cdots<p_{j-1} \\ \mathsf{C}_j(r)}} \frac{N}{p_1\cdots p_{j-1}\log Y(\log N)^{k-0.5}} \ll \frac{N(\log\log N)^{\mathbb{A}}}{(\log N)^{k+0.5}},
\end{aligned}$$

which is acceptable.

Noting that

$$\begin{cases} \dfrac{a_1N}{p_1\cdots p_{j-1}} < p_j \\ p_1\cdots p_{j-1}^2 \leq N \end{cases} \Rightarrow p_{j-1} < p_j,$$

and that $\gamma_j(\ell|\mathbf{p_{j-1}}) = \varpi(\ell)$, we arrive from (4.5)

$$Q_j \doteqdot \sum_{r=0}^{j-1} \sum_{\substack{Y<p_1<\cdots<p_r\le z^{\frac14} \\ Y<p_{r+1}<\cdots<p_{j-1} \\ \mathtt{C}_j(r)}} \sum_{\frac{a_1N}{p_1\cdots p_{j-1}}<p_j\le\frac{2a_1N}{p_1\cdots p_{j-1}}} \frac{1}{\phi(a_1)}\frac{\phi(A)}{A}\frac{4^{k-1}\mathfrak{S}}{(\log z)^{D+\widetilde{D}-2}}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!}\times$$

$$\left\{\sum_{\substack{[\delta_1,\widetilde{\delta}_1]=p_1\cdots p_r \\ \delta_1\le\widetilde{\delta}_1\le z^{\frac14}}}\mu(\delta_1)\mu(\widetilde{\delta}_1)\sum_{i=0}^{D-k}\mathrm{C}_i^{D+\widetilde{D}-k-1}\left(\log\frac{z}{\delta_1^4}\right)^i\left(\log\frac{z}{\widetilde{\delta}_1^4}\right)^{D+\widetilde{D}-k-1-i}\mathrm{C}_{D-k-i}^{D-2-i}(-1)^{D-k-i}\right\}$$
$$+[D\rightleftarrows\widetilde{D}].$$

If $r<j-1$, then by $\mathtt{C}_j(r)$ we have $\log\dfrac{N}{p_1\cdots p_{j-1}}\ge\log p_{j-1}>\log Y\gg\log N$, else if $r=j-1$, by $p_1\cdots p_{j-1}\le z^{\frac24}$ we also have $\log\dfrac{N}{p_1\cdots p_{j-1}}\gg\log N$. Hence, by the prime number theorem, we obtain

$$Q_j \doteqdot \frac{4^{k-1}\mathfrak{S}N}{(\log z)^{D+\widetilde{D}-2}}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!}\sum_{r=0}^{j-1}\sum_{\substack{Y<p_1<\cdots<p_r\le z^{\frac14} \\ Y<p_{r+1}<\cdots<p_{j-1} \\ \mathtt{C}_j(r)}}\frac{1}{p_1\cdots p_{j-1}\log\dfrac{N}{p_1\cdots p_{j-1}}}\times$$

$$\left\{\sum_{\substack{[\delta_1,\widetilde{\delta}_1]=p_1\cdots p_r \\ \delta_1\le\widetilde{\delta}_1\le z^{\frac14}}}\mu(\delta_1)\mu(\widetilde{\delta}_1)\sum_{i=0}^{D-k}\mathrm{C}_i^{D+\widetilde{D}-k-1}\left(\log\frac{z}{\delta_1^4}\right)^i\left(\log\frac{z}{\widetilde{\delta}_1^4}\right)^{D+\widetilde{D}-k-1-i}\mathrm{C}_{D-k-i}^{D-2-i}(-1)^{D-k-i}\right\}$$
$$+[D\rightleftarrows\widetilde{D}].$$

It is possible to obtain a general asymptotic formula for Q_j's here. However, for practical reason, we shall end our discussion now and deal with them case by case.

4.7 Asymptotic formulas for Q_3, Q_4 and Q_5

By the previous discussion, we have

$$Q_3 \doteqdot \frac{4^{k-1}\mathfrak{S}N}{(\log z)^{D+\widetilde{D}-2}}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!}\sum_{(\delta_1,\widetilde{\delta}_1,\pm,f_i,g_i)}\sum_{i=0}^{D-k}\mathrm{C}_i^{D+\widetilde{D}-k-1}\mathrm{C}_{D-k-i}^{D-2-i}(-1)^{D-k-i}$$

$$\left\{\pm\sum_{f_0<p_1\le g_0}\sum_{f_1(p_1)<p_2\le g_1(p_1)}\frac{\left(\log\dfrac{z}{\delta_1^4}\right)^i\left(\log\dfrac{z}{\widetilde{\delta}_1^4}\right)^{D+\widetilde{D}-k-1-i}}{p_1p_2\log\dfrac{N}{p_1p_2}}\right\}+[D\rightleftarrows\widetilde{D}],$$

where the choice of $(\delta_1, \widetilde{\delta}_1, \pm, f_i, g_i)$ can be retrieved from the following table:

	$\delta_1 =$	$\widetilde{\delta}_1 =$	$\pm$	$f_0 =$	$g_0 =$	$f_1(y_1) =$	$g_1(y_1) =$
1^*	1	1	$+$	Y	$N^{\frac{1}{3}}$	y_1	$\sqrt{\frac{N}{y_1}}$
2	1	p_1	$-$	Y	$z^{\frac{1}{4}}$	Y	$\sqrt{\frac{N}{y_1}}$
3^*	p_1	p_1	$+$	Y	$z^{\frac{1}{4}}$	Y	$\sqrt{\frac{N}{y_1}}$
4	1	p_1p_2	$+$	Y	$z^{\frac{1}{8}}$	y_1	$\frac{z^{\frac{1}{4}}}{y_1}$
5	p_1	p_1p_2	$-$	Y	$z^{\frac{1}{8}}$	y_1	$\frac{z^{\frac{1}{4}}}{y_1}$
6	p_2	p_1p_2	$-$	Y	$z^{\frac{1}{8}}$	y_1	$\frac{z^{\frac{1}{4}}}{y_1}$
7	p_1	p_2	$+$	Y	$z^{\frac{1}{4}}$	y_1	$z^{\frac{1}{4}}$
8^*	p_1p_2	p_1p_2	$+$	Y	$z^{\frac{1}{8}}$	y_1	$\frac{z^{\frac{1}{4}}}{y_1}$

The f_i, g_i are found from right to left so that we determine f_1, g_1 before f_0, g_0. Note that we have $\delta_1 = \widetilde{\delta}_1$ in the rows marked by $(*)$, and for each of the remaining rows, we have $\delta_1 < \widetilde{\delta}_1$, so there is a corresponding term in $[D \rightleftarrows \widetilde{D}]$.

To deal with the sum

$$\sum_{f_0 < p_1 \leq g_0} \sum_{f_1(p_1) < p_2 \leq g_1(p_1)} \frac{\left(\log \frac{z}{\delta_1^4}\right)^i \left(\log \frac{z}{\widetilde{\delta}_1^4}\right)^{D+\widetilde{D}-k-1-i}}{p_1 p_2 \log \frac{N}{p_1 p_2}},$$

we first use multinomial theorem to expand the logarithms. Each term in the expansions is of the form

$$a(\log z)^m \sum_{f_0 < p_1 \leq g_0} \sum_{f_1(p_1) < p_2 \leq g_1(p_1)} V_2(p_1, p_2),$$

where

$$V_2(y_1, y_2) = \frac{(\log y_1)^{m_1} (\log y_2)^{m_2}}{y_1 y_2 \log \frac{N}{y_1 y_2}}.$$

We then apply Proposition 2.4.8. Firstly, note that for $f_0 < y_1 \leq g_0$, $f_1(y_1) < y_2 \leq g_1(y_1)$, we have $y_1 y_2 \leq N^{\frac{2}{3}}$, and that the logarithm derivatives of f_i's, g_i's satisfy property(iii) of Proposition 2.4.9. Therefore, by that result, the contributions from R_i's altogether are $\ll \frac{1}{\log^{\mathbb{A}} N}$. Secondly, we can group all of the resulting V_0's by reversing the multinomial expansions.

It follows that the sum is

$$\int_{f_0}^{g_0}\int_{f_1(u_1)}^{g_1(u_1)} \frac{\left(\log\frac{z}{\Theta(\delta_1^4)}\right)^i \left(\log\frac{z}{\Theta(\widetilde{\delta}_1^4)}\right)^{D+\widetilde{D}-k-1-i}}{(u_1\log u_1)(u_2\log u_2)\log\frac{N}{u_1u_2}} du_2du_1 \ + \ \mathcal{O}\left(\frac{1}{\log^{\mathbb{A}}N}\right),$$

where Θ is the function that replaces every p_i in δ_1, $\widetilde{\delta}_1$ by u_i.

We next use the change of variable $u_i = z^{x_i/4}$. This gives

$$Q_3 \ \doteqdot \ \frac{4^{k-1}\mathfrak{S}N}{(\log z)^{k-1}}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!}\sum_{(\Xi,\widetilde{\Xi},\pm,\mathfrak{f}_i,\mathfrak{g}_i)}\sum_{i=0}^{D-k} \mathrm{C}_i^{D+\widetilde{D}-k-1}\mathrm{C}_{D-k-i}^{D-2-i}(-1)^{D-k-i}$$
$$\left\{\pm\int_{\mathfrak{f}_0}^{\mathfrak{g}_0}\int_{\mathfrak{f}_1(x_1)}^{\mathfrak{g}_1(x_1)} \frac{\Xi^i\,\widetilde{\Xi}^{D+\widetilde{D}-k-1-i}}{x_1x_2(\log N-\frac{x_1+x_2}{4}\log z)}dx_2dx_1\right\} + [D \rightleftarrows \widetilde{D}],$$

where the choice of $(\Xi,\widetilde{\Xi},\pm,\mathfrak{f}_i,\mathfrak{g}_i)$ can be retrieved from the following table:

<u>Table 4.7.1</u>

	$\Xi(x_1,x_2)=$	$\widetilde{\Xi}(x_1,x_2)=$	$\pm$	$\mathfrak{f}_0=$	$\mathfrak{g}_0=$	$\mathfrak{f}_1(x_1)=$	$\mathfrak{g}_1(x_1)=$
1^*	1	1	$+$	η	$\frac{4\log N}{3\log z}$	x_1	$\frac{2\log N}{\log z}-\frac{x_1}{2}$
2	1	$1-x_1$	$-$	η	1	η	$\frac{2\log N}{\log z}-\frac{x_1}{2}$
3^*	$1-x_1$	$1-x_1$	$+$	η	1	η	$\frac{2\log N}{\log z}-\frac{x_1}{2}$
4	1	$1-x_1-x_2$	$+$	η	$\frac{1}{2}$	x_1	$1-x_1$
5	$1-x_1$	$1-x_1-x_2$	$-$	η	$\frac{1}{2}$	x_1	$1-x_1$
6	$1-x_2$	$1-x_1-x_2$	$-$	η	$\frac{1}{2}$	x_1	$1-x_1$
7	$1-x_1$	$1-x_2$	$+$	η	1	x_1	1
8^*	$1-x_1-x_2$	$1-x_1-x_2$	$+$	η	$\frac{1}{2}$	x_1	$1-x_1$

Lastly, we would like to replace every $\log z$ that occurs by $\log N$. We have $\log z = \log N - \mathsf{C}\log\log N$. For the ranges of x_1, x_2 under consideration, we have $z^{\frac{x_1}{4}}z^{\frac{x_2}{4}} \le N^{\frac{2}{3}}$, and so $1-\frac{x_1+x_2}{4} > 1-\frac{2+\varepsilon}{3} > 0$. Therefore, if we first replace $(\log N - \frac{x_1+x_2}{4}\log z)$ by $(1-\frac{x_1+x_2}{4})\log N$ in the integrand, then replace $\frac{\log N}{\log z}$ by 1 in Table 4.7.1, and finally replace $(\log z)^{k-1}$ by $(\log N)^{k-1}$, all $\mathcal{O}$-terms produced are acceptable.

In conclusion, we have proved the following theorem:

Theorem 4.7.1

Let $\mathfrak{L}$ be HB-admissible, and let

$$Q_3 := \sum_{N<n\le 2N} \gamma_3(a_1n+b_1)\left(\sum_{\delta_i|a_in+b_i} \lambda^{D}_{\delta_1,\ldots,\delta_k}\right)\left(\sum_{\widetilde{\delta}_i|a_in+b_i} \lambda^{\widetilde{D}}_{\widetilde{\delta}_1,\ldots,\widetilde{\delta}_k}\right),$$

where γ_3 is given by Definition 4.3.1 and $\lambda^{D}_{\delta_1,\ldots,\delta_k}$ is given by Defintion 4.2.1. Suppose D, $\widetilde{D} \ge k+1$, and $\varphi = 4$. Then

$$\begin{aligned} Q_3 \;\doteqdot\;& \frac{4^k\mathfrak{S}N}{(\log N)^k}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!}\sum_{(\Xi,\widetilde{\Xi},\pm,\mathfrak{f}_i,\mathfrak{g}_i)}\;\sum_{[D\rightleftarrows\widetilde{D}]} \\ & \sum_{i=0}^{D-k} \mathrm{C}_i^{D+\widetilde{D}-k-1}\mathrm{C}_{D-k-i}^{D-2-i}(-1)^{D-k-i}\left\{\pm\int_{\mathfrak{f}_0}^{\mathfrak{g}_0}\int_{\mathfrak{f}_1(x_1)}^{\mathfrak{g}_1(x_1)}\frac{\Xi^i\,\widetilde{\Xi}^{D+\widetilde{D}-k-1-i}}{x_1x_2(4-x_1-x_2)}dx_2dx_1\right\}, \end{aligned}$$

where

(i) *the sum over $(\Xi,\widetilde{\Xi},\pm,\mathfrak{f}_i,\mathfrak{g}_i)$ is to refer Table 4.7.1, with every* $\log z$ *replaced by* $\log N$*;*

(ii) *for each $(\Xi,\widetilde{\Xi},\pm,\mathfrak{f}_i,\mathfrak{g}_i)$,*

$$\sum_{[D\rightleftarrows\widetilde{D}]} T_{D,\widetilde{D}} = \begin{cases} T_{D,\widetilde{D}} & \text{if the corresponding row in Table 4.7.1 is marked by } (*); \\ T_{D,\widetilde{D}} + T_{\widetilde{D},D} & \text{otherwise.}\end{cases}$$

We proceed to evaluate Q_4 in a similar way.

Theorem 4.7.2

Let $\mathfrak{L}$ be HB-admissible, and let

$$Q_4 := \sum_{N<n\le 2N} \gamma_4(a_1n+b_1)\left(\sum_{\delta_i|a_in+b_i} \lambda^{D}_{\delta_1,\ldots,\delta_k}\right)\left(\sum_{\widetilde{\delta}_i|a_in+b_i} \lambda^{\widetilde{D}}_{\widetilde{\delta}_1,\ldots,\widetilde{\delta}_k}\right),$$

where γ_4 is given by Definition 4.3.1 and $\lambda^{D}_{\delta_1,\ldots,\delta_k}$ is given by Defintion 4.2.1. Suppose D, $\widetilde{D} \ge k+1$, and $\varphi = 4$. Then

$$\begin{aligned} Q_4 \;\doteqdot\;& \frac{4^k\mathfrak{S}N}{(\log N)^k}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!}\sum_{\substack{(\Xi,\widetilde{\Xi},\pm,\mathfrak{f}_i,\mathfrak{g}_i)\\ [D\rightleftarrows\widetilde{D}]}}\;\sum_{i=0}^{D-k} \mathrm{C}_i^{D+\widetilde{D}-k-1}\mathrm{C}_{D-k-i}^{D-2-i}(-1)^{D-k-i} \\ & \left\{\pm\int_{\mathfrak{f}_0}^{\mathfrak{g}_0}\int_{\mathfrak{f}_1(x_1)}^{\mathfrak{g}_1(x_1)}\int_{\mathfrak{f}_2(x_1,x_2)}^{\mathfrak{g}_2(x_1,x_2)}\frac{\Xi^i\,\widetilde{\Xi}^{D+\widetilde{D}-k-1-i}dx_1dx_2dx_3}{x_1x_2x_3(4-x_1-x_2-x_3)}\right\}, \end{aligned}$$

where the sum over $(\Xi,\widetilde{\Xi},\pm,\mathfrak{f}_i,\mathfrak{g}_i)$ is to refer Table 4.7.2 below, and the sum over $[D \rightleftarrows \widetilde{D}]$ is to follow the convention in Theorem 4.7.1.

Table 4.7.2

	$\Xi(x_1,x_2,x_3)=$	$\widetilde{\Xi}(x_1,x_2,x_3)=$	$\pm$	$\mathfrak{f}_0=$	$\mathfrak{g}_0=$	$\mathfrak{f}_1(x_1)=$	$\mathfrak{g}_1(x_1)=$	$\mathfrak{f}_2(x_1,x_2)=$	$\mathfrak{g}_2(x_1,x_2)=$
1*	1	1	$+$	η	1	x_1	$\frac{4}{3}-\frac{x_1}{3}$	x_2	$2-\frac{x_1}{2}-\frac{x_2}{2}$
2	1	$1-x_1$	$-$	η	1	η	$\frac{4}{3}-\frac{x_1}{3}$	x_2	$2-\frac{x_1}{2}-\frac{x_2}{2}$
3*	$1-x_1$	$1-x_1$	$+$	η	1	η	$\frac{4}{3}-\frac{x_1}{3}$	x_2	$2-\frac{x_1}{2}-\frac{x_2}{2}$
4	1	$1-x_1-x_2$	$+$	η	$\frac{1}{2}$	x_1	$1-x_1$	η	$2-\frac{x_1}{2}-\frac{x_2}{2}$
5	$1-x_1$	$1-x_1-x_2$	$-$	η	$\frac{1}{2}$	x_1	$1-x_1$	η	$2-\frac{x_1}{2}-\frac{x_2}{2}$
6	$1-x_2$	$1-x_1-x_2$	$-$	η	$\frac{1}{2}$	x_1	$1-x_1$	η	$2-\frac{x_1}{2}-\frac{x_2}{2}$
7	$1-x_1$	$1-x_2$	$+$	η	1	x_1	1	η	$2-\frac{x_1}{2}-\frac{x_2}{2}$
8*	$1-x_1-x_2$	$1-x_1-x_2$	$+$	η	$\frac{1}{2}$	x_1	$1-x_1$	η	$2-\frac{x_1}{2}-\frac{x_2}{2}$
9	1	$1-x_1-x_2-x_3$	$-$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$
10	$1-x_1$	$1-x_1-x_2-x_3$	$+$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$
11	$1-x_2$	$1-x_1-x_2-x_3$	$+$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$
12	$1-x_3$	$1-x_1-x_2-x_3$	$+$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$
13	$1-x_1-x_2$	$1-x_1-x_2-x_3$	$-$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$
14	$1-x_1-x_3$	$1-x_1-x_2-x_3$	$-$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$
15	$1-x_2-x_3$	$1-x_1-x_2-x_3$	$-$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$
16	$1-x_1$	$1-x_2-x_3$	$-$	η	$\frac{1}{2}$	x_1	$\frac{1}{2}$	x_2	$1-x_2$
17	$1-x_1-x_2$	$1-x_2-x_3$	$+$	η	$\frac{1}{2}$	x_1	$\frac{1}{2}$	x_2	$1-x_2$
18	$1-x_1-x_3$	$1-x_2-x_3$	$+$	η	$\frac{1}{2}$	x_1	$\frac{1}{2}$	x_2	$1-x_2$
19	$1-x_2$	$1-x_1-x_3$	$-$	η	$\frac{1}{2}$	x_1	$1-x_1$	x_2	$1-x_1$
20	$1-x_1-x_2$	$1-x_1-x_3$	$+$	η	$\frac{1}{2}$	x_1	$1-x_1$	x_2	$1-x_1$
21a	$1-x_3$	$1-x_1-x_2$	$-$	η	$\frac{1}{2}$	x_1	$1-x_1$	x_2	x_1+x_2
21b	$1-x_1-x_2$	$1-x_3$	$-$	η	$\frac{1}{2}$	x_1	$1-x_1$	x_1+x_2	1
22*	$1-x_1-x_2-x_3$	$1-x_1-x_2-x_3$	$+$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$

Proof

The proof is similar to that of Theorem 4.7.1, so we only give a sketch. By writing $\sideset{}{'}\sum_{f_i(\mathbf{p_i})}^{g_i(\mathbf{p_i})}$ to denote the sum $\sum_{f_i(\mathbf{p_i})<p_{i+1}\le g_i(\mathbf{p_i})}$, we have

$$Q_4 \doteqdot \frac{4^{k-1}\mathfrak{S}N}{(\log z)^{D+\widetilde{D}-2}}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!}\sum_{(\delta_1,\widetilde{\delta}_1,\pm,f_i,g_i)}\sum_{i=0}^{D-k}\mathrm{C}_i^{D+\widetilde{D}-k-1}\mathrm{C}_{D-k-i}^{D-2-i}(-1)^{D-k-i}$$
$$\left\{\pm\sideset{}{'}\sum_{f_0(\mathbf{p_0})}^{g_0(\mathbf{p_0})}\sideset{}{'}\sum_{f_1(\mathbf{p_1})}^{g_1(\mathbf{p_1})}\sideset{}{'}\sum_{f_2(\mathbf{p_2})}^{g_2(\mathbf{p_2})}\frac{\left(\log\frac{z}{\delta_1^4}\right)^i\left(\log\frac{z}{\widetilde{\delta}_1^4}\right)^{D+\widetilde{D}-k-1-i}}{p_1p_2p_3\log\frac{N}{p_1p_2p_3}}\right\}+[D\rightleftarrows\widetilde{D}],$$

where we refer to the following table for the sum over $(\delta_1, \widetilde{\delta}_1, \pm, f_i, g_i)$:

	$\delta_1=$	$\widetilde{\delta}_1=$	$\pm$	$f_0=$	$g_0=$	$f_1(y_1)=$	$g_1(y_1)=$	$f_2(y_1,y_2)=$	$g_2(y_1,y_2)=$
1*	1	1	$+$	Y	$N^{\frac{1}{4}}$	y_1	$(\frac{N}{y_1})^{\frac{1}{3}}$	y_2	$\sqrt{\frac{N}{y_1y_2}}$
2	1	p_1	$-$	Y	$z^{\frac{1}{4}}$	Y	$(\frac{N}{y_1})^{\frac{1}{3}}$	y_2	$\sqrt{\frac{N}{y_1y_2}}$
3*	p_1	p_1	$+$	Y	$z^{\frac{1}{4}}$	Y	$(\frac{N}{y_1})^{\frac{1}{3}}$	y_2	$\sqrt{\frac{N}{y_1y_2}}$
4	1	p_1p_2	$+$	Y	$z^{\frac{1}{8}}$	y_1	$\frac{z^{\frac{1}{4}}}{y_1}$	Y	$\sqrt{\frac{N}{y_1y_2}}$
5	p_1	p_1p_2	$-$	Y	$z^{\frac{1}{8}}$	y_1	$\frac{z^{\frac{1}{4}}}{y_1}$	Y	$\sqrt{\frac{N}{y_1y_2}}$
6	p_2	p_1p_2	$-$	Y	$z^{\frac{1}{8}}$	y_1	$\frac{z^{\frac{1}{4}}}{y_1}$	Y	$\sqrt{\frac{N}{y_1y_2}}$
7	p_1	p_2	$+$	Y	$z^{\frac{1}{4}}$	y_1	$z^{\frac{1}{4}}$	Y	$\sqrt{\frac{N}{y_1y_2}}$
8*	p_1p_2	p_1p_2	$+$	Y	$z^{\frac{1}{8}}$	y_1	$\frac{z^{\frac{1}{4}}}{y_1}$	Y	$\sqrt{\frac{N}{y_1y_2}}$
9	1	$p_1p_2p_3$	$-$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{\sqrt{y_1}}$	y_2	$\frac{z^{\frac{1}{4}}}{y_1y_2}$
10	p_1	$p_1p_2p_3$	$+$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{\sqrt{y_1}}$	y_2	$\frac{z^{\frac{1}{4}}}{y_1y_2}$
11	p_2	$p_1p_2p_3$	$+$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{\sqrt{y_1}}$	y_2	$\frac{z^{\frac{1}{4}}}{y_1y_2}$
12	p_3	$p_1p_2p_3$	$+$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{\sqrt{y_1}}$	y_2	$\frac{z^{\frac{1}{4}}}{y_1y_2}$
13	p_1p_2	$p_1p_2p_3$	$-$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{\sqrt{y_1}}$	y_2	$\frac{z^{\frac{1}{4}}}{y_1y_2}$
14	p_1p_3	$p_1p_2p_3$	$-$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{\sqrt{y_1}}$	y_2	$\frac{z^{\frac{1}{4}}}{y_1y_2}$
15	p_2p_3	$p_1p_2p_3$	$-$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{\sqrt{y_1}}$	y_2	$\frac{z^{\frac{1}{4}}}{y_1y_2}$
16	p_1	p_2p_3	$-$	Y	$z^{\frac{1}{8}}$	y_1	$z^{\frac{1}{8}}$	y_2	$\frac{z^{\frac{1}{4}}}{y_2}$
17	p_1p_2	p_2p_3	$+$	Y	$z^{\frac{1}{8}}$	y_1	$z^{\frac{1}{8}}$	y_2	$\frac{z^{\frac{1}{4}}}{y_2}$
18	p_1p_3	p_2p_3	$+$	Y	$z^{\frac{1}{8}}$	y_1	$z^{\frac{1}{8}}$	y_2	$\frac{z^{\frac{1}{4}}}{y_2}$
19	p_2	p_1p_3	$-$	Y	$z^{\frac{1}{8}}$	y_1	$\frac{z^{\frac{1}{4}}}{y_1}$	y_2	$\frac{z^{\frac{1}{4}}}{y_1}$
20	p_1p_2	p_1p_3	$+$	Y	$z^{\frac{1}{8}}$	y_1	$\frac{z^{\frac{1}{4}}}{y_1}$	y_2	$\frac{z^{\frac{1}{4}}}{y_1}$
21a	p_3	p_1p_2	$-$	Y	$z^{\frac{1}{8}}$	y_1	$\frac{z^{\frac{1}{4}}}{y_1}$	y_2	y_1y_2
21b	p_1p_2	p_3	$-$	Y	$z^{\frac{1}{8}}$	y_1	$\frac{z^{\frac{1}{4}}}{y_1}$	y_1y_2	$z^{\frac{1}{4}}$
22*	$p_1p_2p_3$	$p_1p_2p_3$	$+$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{\sqrt{y_1}}$	y_2	$\frac{z^{\frac{1}{4}}}{y_1y_2}$

The f_i, g_i are found from right to left. We have $\delta_1 = \widetilde{\delta}_1$ in the rows marked by $(*)$, and for each of the remaining rows, we have $\delta_1 < \widetilde{\delta}_1$, so there is a corresponding term in $[D \rightleftarrows \widetilde{D}]$. Note that the 21$^{\text{st}}$ row is split into two, since the cases $p_3 < p_1p_2$ and $p_1p_2 < p_3$ are both possible.

Note that for $f_0(\mathbf{y_i}) < y_i \leq g_i(\mathbf{y_i})$, we have $y_1y_2y_3 \leq N^{\frac{3}{4}}$, and that the logarithm derivatives of f_i's, g_i's satisfy property(iii) of Proposition 2.4.9. Therefore, we can again apply Proposition 2.4.8 together with Proposition 2.4.9.

This gives

$$Q_4 \doteqdot \frac{4^{k-1}\mathfrak{S}N}{(\log z)^{D+\widetilde{D}-2}}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!}\sum_{(\delta_1,\widetilde{\delta}_1,\pm,f_i,g_i)}\sum_{i=0}^{D-k}\mathrm{C}_i^{D+\widetilde{D}-k-1}\mathrm{C}_{D-k-i}^{D-2-i}(-1)^{D-k-i}$$
$$\left\{\pm\int_{f_0}^{g_0}\int_{f_1(u_1)}^{g_1(u_1)}\int_{f_2(u_1,u_2)}^{g_2(u_1,u_2)}\frac{\left(\log\frac{z}{\Theta(\delta_1^4)}\right)^i\left(\log\frac{z}{\Theta(\widetilde{\delta}_1^4)}\right)^{D+\widetilde{D}-k-1-i}}{(u_1\log u_1)(u_2\log u_2)(u_3\log u_3)\log\frac{N}{u_1u_2u_3}}du_3du_2du_1\right\}$$
$$+[D\rightleftarrows\widetilde{D}].$$

Finally, we make use of the change of variable $u_i = z^{x_i/4}$ and then replace every z occurred by N. By noting that $1-\frac{x_1+x_2+x_3}{4} > 1-\frac{3+\varepsilon}{4} > 0$, the $\mathcal{O}$-term produced are acceptable. The result follows.

Q.E.D.

We end by evaluating Q_5. As suggested by Lemma 2.3.1, formally there are

$$\frac{3^0+1}{2}+\frac{3^1+1}{2}+\frac{3^2+1}{2}+\frac{3^3+1}{2}+\frac{3^4+1}{2}=63$$

rows in the corresponding table.

<u>Theorem 4.7.3</u>

Let $\mathfrak{L}$ be HB-admissible, and let

$$Q_5 := \sum_{N<n\le 2N}\gamma_5(a_1n+b_1)\left(\sum_{\delta_i|a_in+b_i}\lambda^D_{\delta_1,\ldots,\delta_k}\right)\left(\sum_{\widetilde{\delta}_i|a_in+b_i}\lambda^{\widetilde{D}}_{\widetilde{\delta}_1,\ldots,\widetilde{\delta}_k}\right),$$

where γ_5 is given by Definition 4.3.1 and $\lambda^D_{\delta_1,\ldots,\delta_k}$ is given by Defintion 4.2.1. Suppose D, $\widetilde{D}\ge k+1$, and $\varphi=4$. Then

$$Q_5 \doteqdot \frac{4^k\mathfrak{S}N}{(\log N)^k}\frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!}\sum_{\substack{(\Xi,\widetilde{\Xi},\pm,\mathfrak{f}_i,\mathfrak{g}_i)\\ [D\rightleftarrows\widetilde{D}]}}\sum_{i=0}^{D-k}\mathrm{C}_i^{D+\widetilde{D}-k-1}\mathrm{C}_{D-k-i}^{D-2-i}(-1)^{D-k-i}$$
$$\left\{\pm\int_{\mathfrak{f}_0}^{\mathfrak{g}_0}\int_{\mathfrak{f}_1(\mathbf{x_1})}^{\mathfrak{g}_1(\mathbf{x_1})}\int_{\mathfrak{f}_2(\mathbf{x_2})}^{\mathfrak{g}_2(\mathbf{x_2})}\int_{\mathfrak{f}_3(\mathbf{x_3})}^{\mathfrak{g}_3(\mathbf{x_3})}\frac{\Xi^i\,\widetilde{\Xi}^{D+\widetilde{D}-k-1-i}dx_1dx_2dx_3dx_4}{x_1x_2x_3x_4(4-x_1-x_2-x_3-x_4)}\right\},$$

where the sum over $(\Xi,\widetilde{\Xi},\pm,\mathfrak{f}_i,\mathfrak{g}_i)$ is to refer Table 4.7.3 below, and the sum over $[D\rightleftarrows\widetilde{D}]$ is to follow the convention in Theorem 4.7.1.

Table 4.7.3

	$\Xi(\mathbf{x_3})=$	$\widetilde{\Xi}(\mathbf{x_3})=$	$\pm$	$\mathfrak{f}_0=$	$\mathfrak{g}_0=$	$\mathfrak{f}_1(\mathbf{x_1})=$	$\mathfrak{g}_1(\mathbf{x_1})=$	$\mathfrak{f}_2(\mathbf{x_2})=$	$\mathfrak{g}_2(\mathbf{x_2})=$	$\mathfrak{f}_3(\mathbf{x_3})=$	$\mathfrak{g}_3(\mathbf{x_3})=$
1*	1	1	$+$	η	$\frac{4}{5}$	x_1	$1-\frac{x_1}{4}$	x_2	$\frac{4}{3}-\frac{x_1}{3}-\frac{x_2}{3}$	x_3	$2-\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}$
2	1	$1-x_1$	$-$	η	1	η	$1-\frac{x_1}{4}$	x_2	$\frac{4}{3}-\frac{x_1}{3}-\frac{x_2}{3}$	x_3	$2-\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}$
3*	$1-x_1$	$1-x_1$	$+$	η	1	η	$1-\frac{x_1}{4}$	x_2	$\frac{4}{3}-\frac{x_1}{3}-\frac{x_2}{3}$	x_3	$2-\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}$
4	1	$1-x_1-x_2$	$+$	η	$\frac{1}{2}$	x_1	$1-x_1$	η	$\frac{4}{3}-\frac{x_1}{3}-\frac{x_2}{3}$	x_3	$2-\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}$
5	$1-x_1$	$1-x_1-x_2$	$-$	η	$\frac{1}{2}$	x_1	$1-x_1$	η	$\frac{4}{3}-\frac{x_1}{3}-\frac{x_2}{3}$	x_3	$2-\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}$
6	$1-x_2$	$1-x_1-x_2$	$-$	η	$\frac{1}{2}$	x_1	$1-x_1$	η	$\frac{4}{3}-\frac{x_1}{3}-\frac{x_2}{3}$	x_3	$2-\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}$
7	$1-x_1$	$1-x_2$	$+$	η	1	x_1	1	η	$\frac{4}{3}-\frac{x_1}{3}-\frac{x_2}{3}$	x_3	$2-\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}$
8*	$1-x_1-x_2$	$1-x_1-x_2$	$+$	η	$\frac{1}{2}$	x_1	$1-x_1$	η	$\frac{4}{3}-\frac{x_1}{3}-\frac{x_2}{3}$	x_3	$2-\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}$
9	1	$1-x_1-x_2-x_3$	$-$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$	η	$2-\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}$
10	$1-x_1$	$1-x_1-x_2-x_3$	$+$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$	η	$2-\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}$
11	$1-x_2$	$1-x_1-x_2-x_3$	$+$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$	η	$2-\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}$
12	$1-x_3$	$1-x_1-x_2-x_3$	$+$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$	η	$2-\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}$
13	$1-x_1-x_2$	$1-x_1-x_2-x_3$	$-$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$	η	$2-\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}$
14	$1-x_1-x_3$	$1-x_1-x_2-x_3$	$-$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$	η	$2-\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}$
15	$1-x_2-x_3$	$1-x_1-x_2-x_3$	$-$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$	η	$2-\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}$
16	$1-x_1$	$1-x_2-x_3$	$-$	η	$\frac{1}{2}$	x_1	$\frac{1}{2}$	x_2	$1-x_2$	η	$2-\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}$
17	$1-x_1-x_2$	$1-x_2-x_3$	$+$	η	$\frac{1}{2}$	x_1	$\frac{1}{2}$	x_2	$1-x_2$	η	$2-\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}$
18	$1-x_1-x_3$	$1-x_2-x_3$	$+$	η	$\frac{1}{2}$	x_1	$\frac{1}{2}$	x_2	$1-x_2$	η	$2-\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}$
19	$1-x_2$	$1-x_1-x_3$	$-$	η	$\frac{1}{2}$	x_1	$1-x_1$	x_2	$1-x_1$	η	$2-\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}$
20	$1-x_1-x_2$	$1-x_1-x_3$	$+$	η	$\frac{1}{2}$	x_1	$1-x_1$	x_2	$1-x_1$	η	$2-\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}$
21a	$1-x_3$	$1-x_1-x_2$	$-$	η	$\frac{1}{2}$	x_1	$1-x_1$	x_2	x_1+x_2	η	$2-\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}$
21b	$1-x_1-x_2$	$1-x_3$	$-$	η	$\frac{1}{2}$	x_1	$1-x_1$	x_1+x_2	1	η	$2-\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}$
22*	$1-x_1-x_2-x_3$	$1-x_1-x_2-x_3$	$+$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$	η	$2-\frac{x_1}{2}-\frac{x_2}{2}-\frac{x_3}{2}$

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	$\Xi(\mathbf{x_3})=$	$\widetilde{\Xi}(\mathbf{x_3})=$	$\pm$	$\mathfrak{f}_0=$	$\mathfrak{g}_0=$	$\mathfrak{f}_1(\mathbf{x_1})=$	$\mathfrak{g}_1(\mathbf{x_1})=$	$\mathfrak{f}_2(\mathbf{x_2})=$	$\mathfrak{g}_2(\mathbf{x_2})=$	$\mathfrak{f}_3(\mathbf{x_3})=$	$\mathfrak{g}_3(\mathbf{x_3})=$
23	1	$1-x_1-x_2-x_3-x_4$	$+$	η	$\frac{1}{4}$	x_1	$\frac{1}{3}-\frac{x_1}{3}$	x_2	$\frac{1}{2}-\frac{x_1}{2}-\frac{x_2}{2}$	x_3	$1-x_1-x_2-x_3$
24	$1-x_1$	$1-x_1-x_2-x_3-x_4$	$-$	η	$\frac{1}{4}$	x_1	$\frac{1}{3}-\frac{x_1}{3}$	x_2	$\frac{1}{2}-\frac{x_1}{2}-\frac{x_2}{2}$	x_3	$1-x_1-x_2-x_3$
25	$1-x_2$	$1-x_1-x_2-x_3-x_4$	$-$	η	$\frac{1}{4}$	x_1	$\frac{1}{3}-\frac{x_1}{3}$	x_2	$\frac{1}{2}-\frac{x_1}{2}-\frac{x_2}{2}$	x_3	$1-x_1-x_2-x_3$
26	$1-x_3$	$1-x_1-x_2-x_3-x_4$	$-$	η	$\frac{1}{4}$	x_1	$\frac{1}{3}-\frac{x_1}{3}$	x_2	$\frac{1}{2}-\frac{x_1}{2}-\frac{x_2}{2}$	x_3	$1-x_1-x_2-x_3$
27	$1-x_4$	$1-x_1-x_2-x_3-x_4$	$-$	η	$\frac{1}{4}$	x_1	$\frac{1}{3}-\frac{x_1}{3}$	x_2	$\frac{1}{2}-\frac{x_1}{2}-\frac{x_2}{2}$	x_3	$1-x_1-x_2-x_3$
28	$1-x_1-x_2$	$1-x_1-x_2-x_3-x_4$	$+$	η	$\frac{1}{4}$	x_1	$\frac{1}{3}-\frac{x_1}{3}$	x_2	$\frac{1}{2}-\frac{x_1}{2}-\frac{x_2}{2}$	x_3	$1-x_1-x_2-x_3$
29	$1-x_1-x_3$	$1-x_1-x_2-x_3-x_4$	$+$	η	$\frac{1}{4}$	x_1	$\frac{1}{3}-\frac{x_1}{3}$	x_2	$\frac{1}{2}-\frac{x_1}{2}-\frac{x_2}{2}$	x_3	$1-x_1-x_2-x_3$
30	$1-x_1-x_4$	$1-x_1-x_2-x_3-x_4$	$+$	η	$\frac{1}{4}$	x_1	$\frac{1}{3}-\frac{x_1}{3}$	x_2	$\frac{1}{2}-\frac{x_1}{2}-\frac{x_2}{2}$	x_3	$1-x_1-x_2-x_3$
31	$1-x_2-x_3$	$1-x_1-x_2-x_3-x_4$	$+$	η	$\frac{1}{4}$	x_1	$\frac{1}{3}-\frac{x_1}{3}$	x_2	$\frac{1}{2}-\frac{x_1}{2}-\frac{x_2}{2}$	x_3	$1-x_1-x_2-x_3$
32	$1-x_2-x_4$	$1-x_1-x_2-x_3-x_4$	$+$	η	$\frac{1}{4}$	x_1	$\frac{1}{3}-\frac{x_1}{3}$	x_2	$\frac{1}{2}-\frac{x_1}{2}-\frac{x_2}{2}$	x_3	$1-x_1-x_2-x_3$
33	$1-x_3-x_4$	$1-x_1-x_2-x_3-x_4$	$+$	η	$\frac{1}{4}$	x_1	$\frac{1}{3}-\frac{x_1}{3}$	x_2	$\frac{1}{2}-\frac{x_1}{2}-\frac{x_2}{2}$	x_3	$1-x_1-x_2-x_3$
34	$1-x_1-x_2-x_3$	$1-x_1-x_2-x_3-x_4$	$-$	η	$\frac{1}{4}$	x_1	$\frac{1}{3}-\frac{x_1}{3}$	x_2	$\frac{1}{2}-\frac{x_1}{2}-\frac{x_2}{2}$	x_3	$1-x_1-x_2-x_3$
35	$1-x_1-x_2-x_4$	$1-x_1-x_2-x_3-x_4$	$-$	η	$\frac{1}{4}$	x_1	$\frac{1}{3}-\frac{x_1}{3}$	x_2	$\frac{1}{2}-\frac{x_1}{2}-\frac{x_2}{2}$	x_3	$1-x_1-x_2-x_3$
36	$1-x_1-x_3-x_4$	$1-x_1-x_2-x_3-x_4$	$-$	η	$\frac{1}{4}$	x_1	$\frac{1}{3}-\frac{x_1}{3}$	x_2	$\frac{1}{2}-\frac{x_1}{2}-\frac{x_2}{2}$	x_3	$1-x_1-x_2-x_3$
37	$1-x_2-x_3-x_4$	$1-x_1-x_2-x_3-x_4$	$-$	η	$\frac{1}{4}$	x_1	$\frac{1}{3}-\frac{x_1}{3}$	x_2	$\frac{1}{2}-\frac{x_1}{2}-\frac{x_2}{2}$	x_3	$1-x_1-x_2-x_3$

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	$\Xi(\mathbf{x_3})=$	$\widetilde{\Xi}(\mathbf{x_3})=$	$\pm$	$\mathfrak{f}_0=$	$\mathfrak{g}_0=$	$\mathfrak{f}_1(\mathbf{x_1})=$	$\mathfrak{g}_1(\mathbf{x_1})=$	$\mathfrak{f}_2(\mathbf{x_2})=$	$\mathfrak{g}_2(\mathbf{x_2})=$	$\mathfrak{f}_3(\mathbf{x_3})=$	$\mathfrak{g}_3(\mathbf{x_3})=$
38	$1-x_1$	$1-x_2-x_3-x_4$	$+$	η	$\frac{1}{3}$	x_1	$\frac{1}{3}$	x_2	$\frac{1}{2}-\frac{x_2}{2}$	x_3	$1-x_2-x_3$
39	$1-x_1-x_2$	$1-x_2-x_3-x_4$	$-$	η	$\frac{1}{3}$	x_1	$\frac{1}{3}$	x_2	$\frac{1}{2}-\frac{x_2}{2}$	x_3	$1-x_2-x_3$
40	$1-x_1-x_3$	$1-x_2-x_3-x_4$	$-$	η	$\frac{1}{3}$	x_1	$\frac{1}{3}$	x_2	$\frac{1}{2}-\frac{x_2}{2}$	x_3	$1-x_2-x_3$
41	$1-x_1-x_4$	$1-x_2-x_3-x_4$	$-$	η	$\frac{1}{3}$	x_1	$\frac{1}{3}$	x_2	$\frac{1}{2}-\frac{x_2}{2}$	x_3	$1-x_2-x_3$
42	$1-x_1-x_2-x_3$	$1-x_2-x_3-x_4$	$+$	η	$\frac{1}{3}$	x_1	$\frac{1}{3}$	x_2	$\frac{1}{2}-\frac{x_2}{2}$	x_3	$1-x_2-x_3$
43	$1-x_1-x_2-x_4$	$1-x_2-x_3-x_4$	$+$	η	$\frac{1}{3}$	x_1	$\frac{1}{3}$	x_2	$\frac{1}{2}-\frac{x_2}{2}$	x_3	$1-x_2-x_3$
44	$1-x_1-x_3-x_4$	$1-x_2-x_3-x_4$	$+$	η	$\frac{1}{3}$	x_1	$\frac{1}{3}$	x_2	$\frac{1}{2}-\frac{x_2}{2}$	x_3	$1-x_2-x_3$
45	$1-x_2$	$1-x_1-x_3-x_4$	$+$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$\frac{1}{2}-\frac{x_1}{2}$	x_3	$1-x_1-x_3$
46	$1-x_1-x_2$	$1-x_1-x_3-x_4$	$-$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$\frac{1}{2}-\frac{x_1}{2}$	x_3	$1-x_1-x_3$
47	$1-x_2-x_3$	$1-x_1-x_3-x_4$	$-$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$\frac{1}{2}-\frac{x_1}{2}$	x_3	$1-x_1-x_3$
48	$1-x_2-x_4$	$1-x_1-x_3-x_4$	$-$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$\frac{1}{2}-\frac{x_1}{2}$	x_3	$1-x_1-x_3$
49	$1-x_1-x_2-x_3$	$1-x_1-x_3-x_4$	$+$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$\frac{1}{2}-\frac{x_1}{2}$	x_3	$1-x_1-x_3$
50	$1-x_1-x_2-x_4$	$1-x_1-x_3-x_4$	$+$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$\frac{1}{2}-\frac{x_1}{2}$	x_3	$1-x_1-x_3$

(*continued on the next page*)

	$\Xi(\mathbf{x_3})=$	$\widetilde{\Xi}(\mathbf{x_3})=$	$\pm$	$\mathfrak{f}_0=$	$\mathfrak{g}_0=$	$\mathfrak{f}_1(\mathbf{x_1})=$	$\mathfrak{g}_1(\mathbf{x_1})=$	$\mathfrak{f}_2(\mathbf{x_2})=$	$\mathfrak{g}_2(\mathbf{x_2})=$	$\mathfrak{f}_3(\mathbf{x_3})=$	$\mathfrak{g}_3(\mathbf{x_3})=$
51	$1-x_3$	$1-x_1-x_2-x_4$	$+$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$	x_3	$1-x_1-x_2$
52	$1-x_1-x_3$	$1-x_1-x_2-x_4$	$-$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$	x_3	$1-x_1-x_2$
53	$1-x_2-x_3$	$1-x_1-x_2-x_4$	$-$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$	x_3	$1-x_1-x_2$
54a	$1-x_3-x_4$	$1-x_1-x_2-x_4$	$-$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$\min(1-x_1-x_2,x_1+x_2)$	x_3	$1-x_1-x_2$
54b	$1-x_1-x_2-x_4$	$1-x_3-x_4$	$-$	η	$\frac{1}{4}$	x_1	$\frac{1}{2}-x_1$	x_1+x_2	$\frac{1}{2}$	x_3	$1-x_3$
55	$1-x_1-x_2-x_3$	$1-x_1-x_2-x_4$	$+$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$	x_3	$1-x_1-x_2$
56a	$1-x_4$	$1-x_1-x_2-x_3$	$+$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$	x_3	$x_1+x_2+x_3$
56b	$1-x_1-x_2-x_3$	$1-x_4$	$+$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$	$x_1+x_2+x_3$	1
57a	$1-x_1-x_4$	$1-x_1-x_2-x_3$	$-$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$	x_3	x_2+x_3
57b	$1-x_1-x_2-x_3$	$1-x_1-x_4$	$-$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$	x_2+x_3	$1-x_1$
58a	$1-x_2-x_4$	$1-x_1-x_2-x_3$	$-$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$	x_3	x_1+x_3
58b	$1-x_1-x_2-x_3$	$1-x_2-x_4$	$-$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$1-x_1-x_2$	x_1+x_3	$1-x_2$
59a	$1-x_3-x_4$	$1-x_1-x_2-x_3$	$-$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$\min(1-x_1-x_2,x_1+x_2)$	x_3	x_1+x_2
59b	$1-x_1-x_2-x_3$	$1-x_3-x_4$	$-$	η	$\frac{1}{3}$	x_1	$\frac{1}{2}-\frac{x_1}{2}$	x_2	$\min(1-x_1-x_2,\frac{1}{2})$	$\max(x_3,x_1+x_2)$	$1-x_3$
60	$1-x_1-x_2$	$1-x_3-x_4$	$+$	η	$\frac{1}{2}$	x_1	$\frac{1}{2}$	x_2	$\frac{1}{2}$	x_3	$1-x_3$
61	$1-x_1-x_3$	$1-x_2-x_4$	$+$	η	$\frac{1}{2}$	x_1	$\frac{1}{2}$	x_2	$1-x_2$	x_3	$1-x_2$
62a	$1-x_2-x_3$	$1-x_1-x_4$	$+$	η	$\frac{1}{2}$	x_1	$\frac{1}{2}$	x_2	$1-x_2$	$x_2+x_3-x_1$	$1-x_1$
62b	$1-x_1-x_4$	$1-x_2-x_3$	$+$	η	$\frac{1}{2}$	x_1	$\frac{1}{2}$	x_2	$1-x_2$	x_3	$x_2+x_3-x_1$
63*	$1-x_1-x_2-x_3-x_4$	$1-x_1-x_2-x_3-x_4$	$+$	η	$\frac{1}{4}$	x_1	$\frac{1}{3}-\frac{x_1}{3}$	x_2	$\frac{1}{2}-\frac{x_1}{2}-\frac{x_2}{2}$	x_3	$1-x_1-x_2-x_3$

Proof

The proof is similar to the previous one, but at some point there is little alteration, as we shall see shortly. Now

$$Q_5 \doteqdot \frac{4^{k-1}\mathfrak{S}N}{(\log z)^{D+\widetilde{D}-2}} \frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!} \sum_{(\delta_1,\widetilde{\delta}_1,\pm,f_i,g_i)} \sum_{i=0}^{D-k} \mathrm{C}_i^{D+\widetilde{D}-k-1} \mathrm{C}_{D-k-i}^{D-2-i} (-1)^{D-k-i}$$

$$\left\{ \pm \sideset{}{'}\sum_{f_0(\mathbf{p_0})}^{g_0(\mathbf{p_0})} \sideset{}{'}\sum_{f_1(\mathbf{p_1})}^{g_1(\mathbf{p_1})} \sideset{}{'}\sum_{f_2(\mathbf{p_2})}^{g_2(\mathbf{p_2})} \sideset{}{'}\sum_{f_3(\mathbf{p_3})}^{g_3(\mathbf{p_3})} \frac{\left(\log \frac{z}{\delta_1^4}\right)^i \left(\log \frac{z}{\widetilde{\delta}_1^4}\right)^{D+\widetilde{D}-k-1-i}}{p_1p_2p_3p_4 \log \frac{N}{p_1p_2p_3p_4}} \right\} + [D \rightleftarrows \widetilde{D}],$$

where we refer to the table on the next few pages for the sum over $(\delta_1, \widetilde{\delta}_1, \pm, f_i, g_i)$.

For $f_0(\mathbf{y_i}) < y_i \leq g_i(\mathbf{y_i})$, we have $y_1y_2y_3y_4 \leq N^{\frac{5}{6}}$, see row 7. Note that in a number of rows, the f_i, g_i are not $\mathcal{C}^\infty$ but piecewise $\mathcal{C}^\infty$ functions, due to the involvement of max, min operations. However, we can split any such sums, so that for each of the split sum only $\mathcal{C}^\infty$ functions are involved. Moreover, it is easy to see that their logarithm derivatives satisfy property(iii) of Proposition 2.4.9. Therefore, we can again apply Proposition 2.4.8 together with Proposition 2.4.9. This time not only the multinomial expansions are reversed after that, but also the aforementioned sum-splitting process as well.

This results in

$$Q_5 \doteqdot \frac{4^{k-1}\mathfrak{S}N}{(\log z)^{D+\widetilde{D}-2}} \frac{(D-1)!(\widetilde{D}-1)!}{(D+\widetilde{D}-k-1)!} \sum_{(\delta_1,\widetilde{\delta}_1,\pm,f_i,g_i)} \sum_{i=0}^{D-k} \mathrm{C}_i^{D+\widetilde{D}-k-1} \mathrm{C}_{D-k-i}^{D-2-i} (-1)^{D-k-i}$$

$$\left\{ \pm \int_{f_0}^{g_0} \int_{f_1(u_1)}^{g_1(u_1)} \int_{f_2(\mathbf{u_2})}^{g_2(\mathbf{u_2})} \int_{f_3(\mathbf{u_3})}^{g_3(\mathbf{u_3})} \frac{\left(\log \frac{z}{\Theta(\delta_1^4)}\right)^i \left(\log \frac{z}{\Theta(\widetilde{\delta}_1^4)}\right)^{D+\widetilde{D}-k-1-i} du_4du_3du_2du_1}{(u_1 \log u_1)(u_2 \log u_2)(u_3 \log u_3)(u_4 \log u_4) \log \frac{N}{u_1u_2u_3u_4}} \right\}$$

$$+ [D \rightleftarrows \widetilde{D}].$$

Finally, we make use of the change of variable $u_i = z^{x_i/4}$ and then replace every z occurred by N. By noting that $1 - \frac{x_1+x_2+x_3+x_4}{4} > 1 - \frac{5+\varepsilon}{6} > 0$, the $\mathcal{O}$-term produced are acceptable. The result follows.

Q.E.D.

	$\delta_1=$	$\widetilde{\delta}_1=$	$\pm$	$f_0=$	$g_0=$	$f_1(\mathbf{y_1})=$	$g_1(\mathbf{y_1})=$	$f_2(\mathbf{y_2})=$	$g_2(\mathbf{y_2})=$	$f_3(\mathbf{y_3})=$	$g_3(\mathbf{y_3})=$
1*	1	1	+	Y	$N^{\frac{1}{5}}$	y_1	$(N/y_1)^{\frac{1}{4}}$	y_2	$(N/y_1y_2)^{\frac{1}{3}}$	y_3	$\sqrt{N/y_1y_2y_3}$
2	1	p_1	$-$	Y	$z^{\frac{1}{4}}$	Y	$(N/y_1)^{\frac{1}{4}}$	y_2	$(N/y_1y_2)^{\frac{1}{3}}$	y_3	$\sqrt{N/y_1y_2y_3}$
3*	p_1	p_1	+	Y	$z^{\frac{1}{4}}$	Y	$(N/y_1)^{\frac{1}{4}}$	y_2	$(N/y_1y_2)^{\frac{1}{3}}$	y_3	$\sqrt{N/y_1y_2y_3}$
4	1	p_1p_2	+	Y	$z^{\frac{1}{8}}$	y_1	$z^{\frac{1}{4}}/y_1$	Y	$(N/y_1y_2)^{\frac{1}{3}}$	y_3	$\sqrt{N/y_1y_2y_3}$
5	p_1	p_1p_2	$-$	Y	$z^{\frac{1}{8}}$	y_1	$z^{\frac{1}{4}}/y_1$	Y	$(N/y_1y_2)^{\frac{1}{3}}$	y_3	$\sqrt{N/y_1y_2y_3}$
6	p_2	p_1p_2	$-$	Y	$z^{\frac{1}{8}}$	y_1	$z^{\frac{1}{4}}/y_1$	Y	$(N/y_1y_2)^{\frac{1}{3}}$	y_3	$\sqrt{N/y_1y_2y_3}$
7	p_1	p_2	+	Y	$z^{\frac{1}{4}}$	y_1	$z^{\frac{1}{4}}$	Y	$(N/y_1y_2)^{\frac{1}{3}}$	y_3	$\sqrt{N/y_1y_2y_3}$
8*	p_1p_2	p_1p_2	+	Y	$z^{\frac{1}{8}}$	y_1	$z^{\frac{1}{4}}/y_1$	Y	$(N/y_1y_2)^{\frac{1}{3}}$	y_3	$\sqrt{N/y_1y_2y_3}$
9	1	$p_1p_2p_3$	$-$	Y	$z^{\frac{1}{12}}$	y_1	$z^{\frac{1}{8}}/\sqrt{y_1}$	y_2	$z^{\frac{1}{4}}/y_1y_2$	Y	$\sqrt{N/y_1y_2y_3}$
10	p_1	$p_1p_2p_3$	+	Y	$z^{\frac{1}{12}}$	y_1	$z^{\frac{1}{8}}/\sqrt{y_1}$	y_2	$z^{\frac{1}{4}}/y_1y_2$	Y	$\sqrt{N/y_1y_2y_3}$
11	p_2	$p_1p_2p_3$	+	Y	$z^{\frac{1}{12}}$	y_1	$z^{\frac{1}{8}}/\sqrt{y_1}$	y_2	$z^{\frac{1}{4}}/y_1y_2$	Y	$\sqrt{N/y_1y_2y_3}$
12	p_3	$p_1p_2p_3$	+	Y	$z^{\frac{1}{12}}$	y_1	$z^{\frac{1}{8}}/\sqrt{y_1}$	y_2	$z^{\frac{1}{4}}/y_1y_2$	Y	$\sqrt{N/y_1y_2y_3}$
13	p_1p_2	$p_1p_2p_3$	$-$	Y	$z^{\frac{1}{12}}$	y_1	$z^{\frac{1}{8}}/\sqrt{y_1}$	y_2	$z^{\frac{1}{4}}/y_1y_2$	Y	$\sqrt{N/y_1y_2y_3}$
14	p_1p_3	$p_1p_2p_3$	$-$	Y	$z^{\frac{1}{12}}$	y_1	$z^{\frac{1}{8}}/\sqrt{y_1}$	y_2	$z^{\frac{1}{4}}/y_1y_2$	Y	$\sqrt{N/y_1y_2y_3}$
15	p_2p_3	$p_1p_2p_3$	$-$	Y	$z^{\frac{1}{12}}$	y_1	$z^{\frac{1}{8}}/\sqrt{y_1}$	y_2	$z^{\frac{1}{4}}/y_1y_2$	Y	$\sqrt{N/y_1y_2y_3}$
16	p_1	p_2p_3	$-$	Y	$z^{\frac{1}{8}}$	y_1	$z^{\frac{1}{8}}$	y_2	$z^{\frac{1}{4}}/y_2$	Y	$\sqrt{N/y_1y_2y_3}$
17	p_1p_2	p_2p_3	+	Y	$z^{\frac{1}{8}}$	y_1	$z^{\frac{1}{8}}$	y_2	$z^{\frac{1}{4}}/y_2$	Y	$\sqrt{N/y_1y_2y_3}$
18	p_1p_3	p_2p_3	+	Y	$z^{\frac{1}{8}}$	y_1	$z^{\frac{1}{8}}$	y_2	$z^{\frac{1}{4}}/y_2$	Y	$\sqrt{N/y_1y_2y_3}$
19	p_2	p_1p_3	$-$	Y	$z^{\frac{1}{8}}$	y_1	$z^{\frac{1}{4}}/y_1$	y_2	$z^{\frac{1}{4}}/y_1$	Y	$\sqrt{N/y_1y_2y_3}$
20	p_1p_2	p_1p_3	+	Y	$z^{\frac{1}{8}}$	y_1	$z^{\frac{1}{4}}/y_1$	y_2	$z^{\frac{1}{4}}/y_1$	Y	$\sqrt{N/y_1y_2y_3}$
$21a$	p_3	p_1p_2	$-$	Y	$z^{\frac{1}{8}}$	y_1	$z^{\frac{1}{4}}/y_1$	y_2	y_1y_2	Y	$\sqrt{N/y_1y_2y_3}$
$21b$	p_1p_2	p_3	$-$	Y	$z^{\frac{1}{8}}$	y_1	$z^{\frac{1}{4}}/y_1$	y_1y_2	$z^{\frac{1}{4}}$	Y	$\sqrt{N/y_1y_2y_3}$
22*	$p_1p_2p_3$	$p_1p_2p_3$	+	Y	$z^{\frac{1}{12}}$	y_1	$z^{\frac{1}{8}}/\sqrt{y_1}$	y_2	$z^{\frac{1}{4}}/y_1y_2$	Y	$\sqrt{N/y_1y_2y_3}$

(continued on the next page)

	$\delta_1=$	$\widetilde{\delta}_1=$	$\pm$	$f_0=$	$g_0=$	$f_1(\mathbf{y_1})=$	$g_1(\mathbf{y_1})=$	$f_2(\mathbf{y_2})=$	$g_2(\mathbf{y_2})=$	$f_3(\mathbf{y_3})=$	$g_3(\mathbf{y_3})=$
23	1	$p_1p_2p_3p_4$	$+$	Y	$z^{\frac{1}{16}}$	y_1	$\frac{z^{\frac{1}{12}}}{(y_1)^{1/3}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_1y_2)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_2y_3}$
24	p_1	$p_1p_2p_3p_4$	$-$	Y	$z^{\frac{1}{16}}$	y_1	$\frac{z^{\frac{1}{12}}}{(y_1)^{1/3}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_1y_2)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_2y_3}$
25	p_2	$p_1p_2p_3p_4$	$-$	Y	$z^{\frac{1}{16}}$	y_1	$\frac{z^{\frac{1}{12}}}{(y_1)^{1/3}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_1y_2)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_2y_3}$
26	p_3	$p_1p_2p_3p_4$	$-$	Y	$z^{\frac{1}{16}}$	y_1	$\frac{z^{\frac{1}{12}}}{(y_1)^{1/3}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_1y_2)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_2y_3}$
27	p_4	$p_1p_2p_3p_4$	$-$	Y	$z^{\frac{1}{16}}$	y_1	$\frac{z^{\frac{1}{12}}}{(y_1)^{1/3}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_1y_2)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_2y_3}$
28	p_1p_2	$p_1p_2p_3p_4$	$+$	Y	$z^{\frac{1}{16}}$	y_1	$\frac{z^{\frac{1}{12}}}{(y_1)^{1/3}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_1y_2)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_2y_3}$
29	p_1p_3	$p_1p_2p_3p_4$	$+$	Y	$z^{\frac{1}{16}}$	y_1	$\frac{z^{\frac{1}{12}}}{(y_1)^{1/3}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_1y_2)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_2y_3}$
30	p_1p_4	$p_1p_2p_3p_4$	$+$	Y	$z^{\frac{1}{16}}$	y_1	$\frac{z^{\frac{1}{12}}}{(y_1)^{1/3}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_1y_2)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_2y_3}$
31	p_2p_3	$p_1p_2p_3p_4$	$+$	Y	$z^{\frac{1}{16}}$	y_1	$\frac{z^{\frac{1}{12}}}{(y_1)^{1/3}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_1y_2)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_2y_3}$
32	p_2p_4	$p_1p_2p_3p_4$	$+$	Y	$z^{\frac{1}{16}}$	y_1	$\frac{z^{\frac{1}{12}}}{(y_1)^{1/3}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_1y_2)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_2y_3}$
33	p_3p_4	$p_1p_2p_3p_4$	$+$	Y	$z^{\frac{1}{16}}$	y_1	$\frac{z^{\frac{1}{12}}}{(y_1)^{1/3}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_1y_2)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_2y_3}$
34	$p_1p_2p_3$	$p_1p_2p_3p_4$	$-$	Y	$z^{\frac{1}{16}}$	y_1	$\frac{z^{\frac{1}{12}}}{(y_1)^{1/3}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_1y_2)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_2y_3}$
35	$p_1p_2p_4$	$p_1p_2p_3p_4$	$-$	Y	$z^{\frac{1}{16}}$	y_1	$\frac{z^{\frac{1}{12}}}{(y_1)^{1/3}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_1y_2)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_2y_3}$
36	$p_1p_3p_4$	$p_1p_2p_3p_4$	$-$	Y	$z^{\frac{1}{16}}$	y_1	$\frac{z^{\frac{1}{12}}}{(y_1)^{1/3}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_1y_2)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_2y_3}$
37	$p_2p_3p_4$	$p_1p_2p_3p_4$	$-$	Y	$z^{\frac{1}{16}}$	y_1	$\frac{z^{\frac{1}{12}}}{(y_1)^{1/3}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_1y_2)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_2y_3}$

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	$\delta_1=$	$\widetilde{\delta}_1=$	$\pm$	$f_0=$	$g_0=$	$f_1(\mathbf{y_1})=$	$g_1(\mathbf{y_1})=$	$f_2(\mathbf{y_2})=$	$g_2(\mathbf{y_2})=$	$f_3(\mathbf{y_3})=$	$g_3(\mathbf{y_3})=$
38	p_1	$p_2p_3p_4$	$+$	Y	$z^{\frac{1}{12}}$	y_1	$z^{\frac{1}{12}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_2)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_2y_3}$
39	p_1p_2	$p_2p_3p_4$	$-$	Y	$z^{\frac{1}{12}}$	y_1	$z^{\frac{1}{12}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_2)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_2y_3}$
40	p_1p_3	$p_2p_3p_4$	$-$	Y	$z^{\frac{1}{12}}$	y_1	$z^{\frac{1}{12}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_2)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_2y_3}$
41	p_1p_4	$p_2p_3p_4$	$-$	Y	$z^{\frac{1}{12}}$	y_1	$z^{\frac{1}{12}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_2)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_2y_3}$
42	$p_1p_2p_3$	$p_2p_3p_4$	$+$	Y	$z^{\frac{1}{12}}$	y_1	$z^{\frac{1}{12}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_2)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_2y_3}$
43	$p_1p_2p_4$	$p_2p_3p_4$	$+$	Y	$z^{\frac{1}{12}}$	y_1	$z^{\frac{1}{12}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_2)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_2y_3}$
44	$p_1p_3p_4$	$p_2p_3p_4$	$+$	Y	$z^{\frac{1}{12}}$	y_1	$z^{\frac{1}{12}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_2)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_2y_3}$
45	p_2	$p_1p_3p_4$	$+$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{(y_1)^{1/2}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_1)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_3}$
46	p_1p_2	$p_1p_3p_4$	$-$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{(y_1)^{1/2}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_1)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_3}$
47	p_2p_3	$p_1p_3p_4$	$-$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{(y_1)^{1/2}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_1)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_3}$
48	p_2p_4	$p_1p_3p_4$	$-$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{(y_1)^{1/2}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_1)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_3}$
49	$p_1p_2p_3$	$p_1p_3p_4$	$+$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{(y_1)^{1/2}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_1)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_3}$
50	$p_1p_2p_4$	$p_1p_3p_4$	$+$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{(y_1)^{1/2}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_1)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_3}$

(*continued on the next page*)

	$\delta_1=$	$\widetilde{\delta}_1=$	$\pm$	$f_0=$	$g_0=$	$f_1(\mathbf{y_1})=$	$g_1(\mathbf{y_1})=$	$f_2(\mathbf{y_2})=$	$g_2(\mathbf{y_2})=$	$f_3(\mathbf{y_3})=$	$g_3(\mathbf{y_3})=$
51	p_3	$p_1p_2p_4$	$+$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{(y_1)^{1/2}}$	y_2	$\frac{z^{\frac{1}{4}}}{y_1y_2}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_2}$
52	p_1p_3	$p_1p_2p_4$	$-$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{(y_1)^{1/2}}$	y_2	$\frac{z^{\frac{1}{4}}}{y_1y_2}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_2}$
53	p_2p_3	$p_1p_2p_4$	$-$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{(y_1)^{1/2}}$	y_2	$\frac{z^{\frac{1}{4}}}{y_1y_2}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_2}$
54a	p_3p_4	$p_1p_2p_4$	$-$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{(y_1)^{1/2}}$	y_2	$\min(\frac{z^{\frac{1}{4}}}{y_1y_2}, y_1y_2)$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_2}$
54b	$p_1p_2p_4$	p_3p_4	$-$	Y	$z^{\frac{1}{16}}$	y_1	$\frac{z^{\frac{1}{8}}}{y_1}$	y_1y_2	$z^{\frac{1}{8}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_3}$
55	$p_1p_2p_3$	$p_1p_2p_4$	$+$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{(y_1)^{1/2}}$	y_2	$\frac{z^{\frac{1}{4}}}{y_1y_2}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_2}$
56a	p_4	$p_1p_2p_3$	$+$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{(y_1)^{1/2}}$	y_2	$\frac{z^{\frac{1}{4}}}{y_1y_2}$	y_3	$y_1y_2y_3$
56b	$p_1p_2p_3$	p_4	$+$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{(y_1)^{1/2}}$	y_2	$\frac{z^{\frac{1}{4}}}{y_1y_2}$	$y_1y_2y_3$	$z^{\frac{1}{4}}$
57a	p_1p_4	$p_1p_2p_3$	$-$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{(y_1)^{1/2}}$	y_2	$\frac{z^{\frac{1}{4}}}{y_1y_2}$	y_3	y_2y_3
57b	$p_1p_2p_3$	p_1p_4	$-$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{(y_1)^{1/2}}$	y_2	$\frac{z^{\frac{1}{4}}}{y_1y_2}$	y_2y_3	$\frac{z^{\frac{1}{4}}}{y_1}$
58a	p_2p_4	$p_1p_2p_3$	$-$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{(y_1)^{1/2}}$	y_2	$\frac{z^{\frac{1}{4}}}{y_1y_2}$	y_3	y_1y_3
58b	$p_1p_2p_3$	p_2p_4	$-$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{(y_1)^{1/2}}$	y_2	$\frac{z^{\frac{1}{4}}}{y_1y_2}$	y_1y_3	$\frac{z^{\frac{1}{4}}}{y_2}$
59a	p_3p_4	$p_1p_2p_3$	$-$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{(y_1)^{1/2}}$	y_2	$\min(\frac{z^{\frac{1}{4}}}{y_1y_2}, y_1y_2)$	y_3	y_1y_2
59b	$p_1p_2p_3$	p_3p_4	$-$	Y	$z^{\frac{1}{12}}$	y_1	$\frac{z^{\frac{1}{8}}}{(y_1)^{1/2}}$	y_2	$\min(\frac{z^{\frac{1}{4}}}{y_1y_2}, z^{\frac{1}{8}})$	$\max(y_3, y_1y_2)$	$\frac{z^{\frac{1}{4}}}{y_3}$
60	p_1p_2	p_3p_4	$+$	Y	$z^{\frac{1}{8}}$	y_1	$z^{\frac{1}{8}}$	y_2	$z^{\frac{1}{8}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_3}$
61	p_1p_3	p_2p_4	$+$	Y	$z^{\frac{1}{8}}$	y_1	$z^{\frac{1}{8}}$	y_2	$\frac{z^{\frac{1}{4}}}{y_2}$	y_3	$\frac{z^{\frac{1}{4}}}{y_2}$
62a	p_2p_3	p_1p_4	$+$	Y	$z^{\frac{1}{8}}$	y_1	$z^{\frac{1}{8}}$	y_2	$\frac{z^{\frac{1}{4}}}{y_2}$	$\frac{y_2y_3}{y_1}$	$\frac{z^{\frac{1}{4}}}{y_1}$
62b	p_1p_4	p_2p_3	$+$	Y	$z^{\frac{1}{8}}$	y_1	$z^{\frac{1}{8}}$	y_2	$\frac{z^{\frac{1}{4}}}{y_2}$	y_3	$\frac{y_2y_3}{y_1}$
63*	$p_1p_2p_3p_4$	$p_1p_2p_3p_4$	$+$	Y	$z^{\frac{1}{16}}$	y_1	$\frac{z^{\frac{1}{12}}}{(y_1)^{1/3}}$	y_2	$\frac{z^{\frac{1}{8}}}{(y_1y_2)^{1/2}}$	y_3	$\frac{z^{\frac{1}{4}}}{y_1y_2y_3}$

4.8 Conclusion

We shall fix $\varphi = 4$, $\eta = \frac{1}{1024}$ and prove Theorem 1.2.2 and Theorem 1.2.3 in this section.

Let us define some notations. Let $\alpha_i = \alpha_i(n) := \omega(a_i n + b_i)$. For $0 \le j \le 5$, let $v_{k,j}(m, n)$ be such that

$$Q_j(D, \widetilde{D}) \doteqdot \frac{4^k \mathfrak{S} N}{(\log N)^k} \cdot v_{k,j}(D-k, \widetilde{D}-k).$$

We shall apply MATLAB to find $v_{k,j}(m, n)$ $(2 \le j \le 5)$ by numerical integrations. The code used is included in Appendix A.

We let $V_{k,j}(m)$ be the $m{\times}m$ matrix

$$\begin{pmatrix} v_{k,j}(1,1) & \cdots & v_{k,j}(1,m) \\ \vdots & & \vdots \\ v_{k,j}(m,1) & \cdots & v_{k,j}(m,m) \end{pmatrix}.$$

Appendix B contains the $V_{k,j}(m)$'s that are used in the proof below.

Finally, we shall write Λ_k^D for $\displaystyle\sum_{\substack{\delta_i | a_i n + b_i \\ (1 \le i \le k)}} \lambda^D_{\delta_1, \ldots, \delta_k}$.

Proof of Theorem 1.2.2

(a) It suffices to show that there exist w_1, w_2 such that

$$\sum_{N < n \le 2N} \left[\begin{array}{ccc} & & \gamma_3(a_2 n + b_2) \\ & & + \\ & & \gamma_2(a_2 n + b_2) \; -1 \\ & & + \\ \gamma_1(a_1 n + b_1) & + & \gamma_1(a_2 n + b_2) \end{array} \right] \left(w_1 \Lambda_2^3 + w_2 \Lambda_2^4\right)^2$$

is positive for all arbitrarily large N.

By the established results, we see that it suffices to find $\mathbf{w_2}$ such that

$$\mathbf{w_2}^T [2V_{2,1} + V_{2,2} + V_{2,3} - V_{2,0}] \mathbf{w_2} > 0,$$

where all $V_{2,j}$'s are two by two matrices.

Let M be the resulting matrix in the square brackets. By the same idea used in the proof of Theorem 1.2.1 (Section 3.4), we try to solve

$$\begin{pmatrix} M & & -1 \\ & & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \ell \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since the solution is approximately equal to

$$\begin{pmatrix} w_1 \\ w_2 \\ \ell \end{pmatrix} = \begin{pmatrix} 0.4359 \\ 0.5641 \\ 0.0359 \end{pmatrix},$$

the result follows.

(b) It suffices to show that there exist w_1, w_2 such that

$$\sum_{N<n\leq 2N} \begin{bmatrix} \gamma_3(a_1n+b_1) & + & \gamma_3(a_2n+b_2) & & & \\ + & & + & & & \\ \gamma_2(a_1n+b_1) & + & \gamma_2(a_2n+b_2) & + & \gamma_2(a_3n+b_3) & -2 \\ + & & + & & + & \\ \gamma_1(a_1n+b_1) & + & \gamma_1(a_2n+b_2) & + & \gamma_1(a_3n+b_3) & \end{bmatrix} \left(w_1\Lambda_3^4 + w_2\Lambda_3^5\right)^2$$

is positive for all arbitrarily large N.

It in turn suffices to find $\mathbf{w_2}$ such that

$$\mathbf{w_2}^T[3V_{3,1} + 3V_{3,2} + 2V_{3,3} - 2V_{3,0}]\mathbf{w_2} > 0,$$

where all $V_{3,j}$'s are two by two matrices.

Let M be the resulting matrix in the square brackets. We try to solve

$$\begin{pmatrix} \begin{matrix} & M & \end{matrix} & \begin{matrix} -1 \\ -1 \end{matrix} \\ \begin{matrix} 1 & 1 \end{matrix} & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \ell \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since the solution is approximately equal to

$$\begin{pmatrix} w_1 \\ w_2 \\ \ell \end{pmatrix} = \begin{pmatrix} 0.3352 \\ 0.6648 \\ 0.2406 \end{pmatrix},$$

the result follows.

(c) Now

$$\sum_{N<n\leq 2N} \begin{bmatrix} \frac{12}{15}\gamma_4(a_1n+b_1) & + & \frac{12}{15}\gamma_4(a_2n+b_2) & + & \frac{12}{15}\gamma_4(a_3n+b_3) & + & \frac{12}{15}\gamma_4(a_4n+b_4) & \\ + & & + & & + & & + & \\ \frac{16}{15}\gamma_3(a_1n+b_1) & + & \frac{16}{15}\gamma_3(a_2n+b_2) & + & \frac{16}{15}\gamma_3(a_3n+b_3) & + & \frac{16}{15}\gamma_3(a_4n+b_4) & \\ + & & + & & + & & + & -4 \\ \frac{20}{15}\gamma_2(a_1n+b_1) & + & \frac{20}{15}\gamma_2(a_2n+b_2) & + & \frac{20}{15}\gamma_2(a_3n+b_3) & + & \frac{20}{15}\gamma_2(a_4n+b_4) & \\ + & & + & & + & & + & \\ \frac{20}{15}\gamma_1(a_1n+b_1) & + & \frac{20}{15}\gamma_1(a_2n+b_2) & + & \frac{20}{15}\gamma_1(a_3n+b_3) & + & \frac{20}{15}\gamma_1(a_4n+b_4) & \end{bmatrix} \left(\Lambda_4^6\right)^2$$

is positive for all arbitrarily large N. It follows that there exist infinitely many n such that the outcome in the square brackets is positive. For any

such n, if we have $\gamma_j(a_1n+b_1) = 0$ for all $1 \le j \le 4$, then the outcome in the square brackets is
$$\le 3 \times \frac{20}{15} - 4 = 0.$$
Hence, we must have $\gamma_1(a_in+b_i) + \gamma_2(a_in+b_i) + \gamma_3(a_in+b_i) + \gamma_4(a_in+b_i) = 1$ for all $1 \le i \le 4$. On the other hand, for this n, we have
$$\frac{4}{15}(7-\alpha_1) + \frac{4}{15}(7-\alpha_2) + \frac{4}{15}(7-\alpha_3) + \frac{4}{15}(7-\alpha_4) > 4,$$
and so $13 > \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$. This shows $\lceil 4 : 12\,, 4 \rfloor$.

(d) It suffices to show that there exist w_1, w_2 such that
$$\sum_{N<n\le 2N} \left[\begin{array}{c} \sum_{i=1}^{4} \frac{1}{3}\gamma_5(a_in+b_i) \\ + \\ \sum_{i=1}^{4} \frac{2}{3}\gamma_4(a_in+b_i) \\ + \\ \sum_{i=1}^{4} \frac{3}{3}\gamma_3(a_in+b_i) \\ + \\ \sum_{i=1}^{4} \frac{4}{3}\gamma_2(a_in+b_i) \\ + \\ \sum_{i=1}^{4} \frac{4}{3}\gamma_1(a_in+b_i) \end{array} - 4 \right] \left(w_1\Lambda_4^5 + w_2\Lambda_4^6\right)^2 > 0$$
for all arbitrarily large N, because
$$\sum_{i=1}^{4} \frac{1}{3}(6-\alpha_i) > 4 \ \Rightarrow\ 12 > \sum_{i=1}^{4} \alpha_i.$$
It in turn suffices to find $\mathbf{w_2}$ such that
$$\mathbf{w_2}^T M \mathbf{w_2} > 0,$$
where M is the two by two matrix
$$\frac{4}{3}V_{4,1} + \frac{4}{3}V_{4,2} + \frac{3}{3}V_{4,3} + \frac{2}{3}V_{4,4} + \frac{1}{3}V_{4,5} - V_{4,0}.$$
We try to solve
$$\begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{M}} & -1 \\ & & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \ell \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$
Since the solution is approximately equal to
$$\begin{pmatrix} w_1 \\ w_2 \\ \ell \end{pmatrix} = \begin{pmatrix} 0.2126 \\ 0.7874 \\ 0.2178 \end{pmatrix},$$
the result follows.

(e) It suffices to show that there exist w_1, w_2, w_3 such that

$$\sum_{N<n\le 2N}\left[\begin{array}{c}\sum_{i=1}^{5}\frac{35}{44}\gamma_5(a_in+b_i)\\ +\\ \sum_{i=1}^{5}\frac{40}{44}\gamma_4(a_in+b_i)\\ +\\ \sum_{i=1}^{5}\frac{45}{44}\gamma_3(a_in+b_i)\\ +\\ \sum_{i=1}^{5}\frac{50}{44}\gamma_2(a_in+b_i)\\ +\\ \sum_{i=1}^{5}\frac{55}{44}\gamma_1(a_in+b_i)\end{array} -5\right]\left(w_1\Lambda_5^6+w_2\Lambda_5^7+w_3\Lambda_5^8\right)^2>0$$

for all arbitrarily large N, because

$$\sum_{i=1}^{5}\frac{5}{44}(12-\alpha_i)>5 \;\Rightarrow\; 16>\sum_{i=1}^{5}\alpha_i.$$

It in turn suffices to find $\mathbf{w_3}$ such that

$$\mathbf{w_3}^T M\mathbf{w_3}>0,$$

where M is the three by three matrix

$$\frac{55}{44}V_{5,1}+\frac{50}{44}V_{5,2}+\frac{45}{44}V_{5,3}+\frac{40}{44}V_{5,4}+\frac{35}{44}V_{5,5}-V_{5,0}.$$

We try to solve

$$\begin{pmatrix} & & & -1\\ & M & & -1\\ & & & -1\\ 1 & 1 & 1 & 0\end{pmatrix}\begin{pmatrix}w_1\\ w_2\\ w_3\\ \ell\end{pmatrix}=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

Since the solution is approximately equal to

$$\begin{pmatrix}w_1\\ w_2\\ w_3\\ \ell\end{pmatrix}=\begin{pmatrix}0.2639\\0.3065\\0.4296\\0.8428\end{pmatrix},$$

the result follows.

Q.E.D.

We finish by indicating how to obtain Theorem 1.2.3. Observe that this is related to the weight

$$\left[\begin{array}{ccccccccc}
\frac{4}{3}\gamma_5(a_1n+b_1) & & & & & & & & \\
+ & & & & & & & & \\
\frac{4}{3}\gamma_4(a_1n+b_1) & + & \frac{4}{3}\gamma_4(a_2n+b_2) & & & & & & \\
+ & & + & & & & & & \\
\frac{4}{3}\gamma_3(a_1n+b_1) & + & \frac{4}{3}\gamma_3(a_2n+b_2) & + & \frac{4}{3}\gamma_3(a_3n+b_3) & & & & \\
+ & & + & & + & & & & \\
\frac{4}{3}\gamma_2(a_1n+b_1) & + & \frac{4}{3}\gamma_2(a_2n+b_2) & + & \frac{4}{3}\gamma_2(a_3n+b_3) & + & \frac{4}{3}\gamma_2(a_4n+b_4) & & \\
+ & & + & & + & & + & & \\
\frac{4}{3}\gamma_1(a_1n+b_1) & + & \frac{4}{3}\gamma_1(a_2n+b_2) & + & \frac{4}{3}\gamma_1(a_3n+b_3) & + & \frac{4}{3}\gamma_1(a_4n+b_4) & - & 4
\end{array}\right].$$

In view of the weight used in part (d) of the previous proof, the result follows.

Appendix A

MATLAB code for $v_{k,j}(m,n)$

A.1 Code for $v_{k,2}(m,n)$

```
function Q=Q2(k,d,db,eta)
    f=@(x)1./(4-x)./x;

    g1=@(x,ii)(1-x).^(d+db-k-1-ii).*f(x);
    x1max=1;
    x1min=eta;
    g1l=@(x)0;
    for ii=0:1:d-k
        g1l=@(x)g1l(x)+g1(x,ii)*nchoosek(d+db-k-1,ii)*nchoosek(d...
            -2-ii,d-k-ii)*(-1)^(d-k-ii);
    end
    g1r=@(x)0;
    for ii=0:1:db-k
        g1r=@(x)g1r(x)+g1(x,ii)*nchoosek(d+db-k-1,ii)*nchoosek(db...
            -2-ii,db-k-ii)*(-1)^(db-k-ii);
    end
    G1=@(x)g1l(x)+g1r(x);
    J1=integral(G1,x1min,x1max);

    J2=0;
        g2=@(x)(1-x).^(d+db-k-1).*f(x);
        x2max=1;
        x2min=eta;
        J2=integral(g2,x2min,x2max);

        g2=@(x)f(x);
        x2max=2;
        x2min=eta;
        J2=J2+integral(g2,x2min,x2max);
    J2=J2*nchoosek(d+db-2*k,d-k);

    Q=-J1+J2;
    Q=Q*factorial(d-1)*factorial(db-1)/factorial(d+db-k-1);
```

A.2 Code for $v_{k,3}(m,n)$

```
function Q=Q3(k,d,db,eta)
    f=@(x,y)1./(4-x-y)./x./y;

    %%%%J1: row 2
    g1=@(x,y,ii)(1-x).^(d+db-k-1-ii).*f(x,y);
    x1max=1; y1max=@(x)2-x/2;
    x1min=eta; y1min=eta;
    g1l=@(x,y)0;
    for ii=0:1:d-k
        g1l=@(x,y)g1l(x,y)+g1(x,y,ii)*nchoosek(d+db-k-1,ii)*...
            nchoosek(d-2-ii,d-k-ii)*(-1)^(d-k-ii);
    end
    g1r=@(x,y)0;
    for ii=0:1:db-k
        g1r=@(x,y)g1r(x,y)+g1(x,y,ii)*nchoosek(d+db-k-1,ii)*...
            nchoosek(db-2-ii,db-k-ii)*(-1)^(db-k-ii);
    end
    G1=@(x,y)g1l(x,y)+g1r(x,y);
    J1=integral2(G1,x1min,x1max,y1min,y1max);

    %%%%J2: row 1,3
    J2=0;
        g2=@(x,y)(1-x).^(d+db-k-1).*f(x,y);
        x2max=1; y2max=@(x)2-x/2;
        x2min=eta; y2min=eta;
        J2=integral2(g2,x2min,x2max,y2min,y2max);

        g2=@(x,y)f(x,y);
        x2max=4/3; y2max=@(x)2-x/2;
        x2min=eta; y2min=@(x)x;
        J2=J2+integral2(g2,x2min,x2max,y2min,y2max);
    J2=J2*nchoosek(d+db-2*k,d-k);

    %%%%J3: row 4-6
    g3=@(x,y,ii)(1-(1-x).^(ii)-(1-y).^(ii)).*(1-x-y).^(d+db-k-1-...
        ii).*f(x,y);
    x3max=1/2; y3max=@(x)1-x;
    x3min=eta; y3min=@(x)x;
    g3l=@(x,y)0;
    for ii=0:1:d-k
        g3l=@(x,y)g3l(x,y)+g3(x,y,ii)*nchoosek(d+db-k-1,ii)*...
            nchoosek(d-2-ii,d-k-ii)*(-1)^(d-k-ii);
    end
    g3r=@(x,y)0;
    for ii=0:1:db-k
        g3r=@(x,y)g3r(x,y)+g3(x,y,ii)*nchoosek(d+db-k-1,ii)*...
            nchoosek(db-2-ii,db-k-ii)*(-1)^(db-k-ii);
    end
    G3=@(x,y)g3l(x,y)+g3r(x,y);
    J3=integral2(G3,x3min,x3max,y3min,y3max);
```

```
    %%%%J6: row 7
    g6=@(x,y,ii)(1-x).^(ii).*(1-y).^(d+db-k-1-ii).*f(x,y);
    x6max=1; y6max=1;
    x6min=eta; y6min=@(x)x;
    g6l=@(x,y)0;
    for ii=0:1:d-k
        g6l=@(x,y)g6l(x,y)+g6(x,y,ii)*nchoosek(d+db-k-1,ii)*...
            nchoosek(d-2-ii,d-k-ii)*(-1)^(d-k-ii);
    end
    g6r=@(x,y)0;
    for ii=0:1:db-k
        g6r=@(x,y)g6r(x,y)+g6(x,y,ii)*nchoosek(d+db-k-1,ii)*...
            nchoosek(db-2-ii,db-k-ii)*(-1)^(db-k-ii);
    end
    G6=@(x,y)g6l(x,y)+g6r(x,y);
    J6=integral2(G6,x6min,x6max,y6min,y6max);

    %%%%J7: row 8
    J7=0;
        g7=@(x,y)(1-x-y).^(d+db-k-1).*f(x,y);
        x7max=1/2; y7max=@(x)1-x;
        x7min=eta; y7min=@(x)x;
        J7=integral2(g7,x7min,x7max,y7min,y7max);
    J7=J7*nchoosek(d+db-2*k,d-k);

    Q=-J1+J2+J3+J6+J7;
    Q=Q*factorial(d-1)*factorial(db-1)/factorial(d+db-k-1);
```

A.3 Code for $v_{k,4}(m,n)$

```
function Q=Q4(k,d,db,eta)
    f=@(x,y,z)1./(4-x-y-z)./x./y./z;

    %%%%J1: row 2
    g1=@(x,y,z,ii)(1-x).^(d+db-k-1-ii).*f(x,y,z);
    x1max=1;     y1max=@(x)4/3-x/3;  z1max=@(x,y)2-x/2-y/2;
    x1min=eta;  y1min=eta;          z1min=@(x,y)y;
    g1l=@(x,y,z)0;
    for ii=0:1:d-k
        g1l=@(x,y,z)g1l(x,y,z)+g1(x,y,z,ii)*nchoosek(d+db-k-1,ii)...
            *nchoosek(d-2-ii,d-k-ii)*(-1)^(d-k-ii);
    end
    g1r=@(x,y,z)0;
    for ii=0:1:db-k
        g1r=@(x,y,z)g1r(x,y,z)+g1(x,y,z,ii)*nchoosek(d+db-k-1,ii)...
            *nchoosek(db-2-ii,db-k-ii)*(-1)^(db-k-ii);
    end
    G1=@(x,y,z)g1l(x,y,z)+g1r(x,y,z);
    J1=integral3(G1,x1min,x1max,y1min,y1max,z1min,z1max);

    %%%%J2: row 1,3
    J2=0;
        g2=@(x,y,z)(1-x).^(d+db-k-1).*f(x,y,z);
        x2max=1;     y2max=@(x)4/3-x/3;  z2max=@(x,y)2-x/2-y/2;
        x2min=eta;  y2min=eta;          z2min=@(x,y)y;
        J2=integral3(g2,x2min,x2max,y2min,y2max,z2min,z2max);

        g2=@(x,y,z)f(x,y,z);
        x2max=1;     y2max=@(x)4/3-x/3;  z2max=@(x,y)2-x/2-y/2;
        x2min=eta;  y2min=@(x)x;        z2min=@(x,y)y;
        J2=J2+integral3(g2,x2min,x2max,y2min,y2max,z2min,z2max);
    J2=J2*nchoosek(d+db-2*k,d-k);

    %%%%J3: row 4-6
    g3=@(x,y,z,ii)(1-(1-x).^(ii)-(1-y).^(ii)).*(1-x-y).^(d+db-k...
       -1-ii).*f(x,y,z);
    x3max=1/2; y3max=@(x)1-x;    z3max=@(x,y)2-x/2-y/2;
    x3min=eta; y3min=@(x)x;      z3min=eta;
    g3l=@(x,y,z)0;
    for ii=0:1:d-k
        g3l=@(x,y,z)g3l(x,y,z)+g3(x,y,z,ii)*nchoosek(d+db-k-1,ii)...
            *nchoosek(d-2-ii,d-k-ii)*(-1)^(d-k-ii);
    end
    g3r=@(x,y,z)0;
    for ii=0:1:db-k
        g3r=@(x,y,z)g3r(x,y,z)+g3(x,y,z,ii)*nchoosek(d+db-k-1,ii)...
            *nchoosek(db-2-ii,db-k-ii)*(-1)^(db-k-ii);
    end
    G3=@(x,y,z)g3l(x,y,z)+g3r(x,y,z);
    J3=integral3(G3,x3min,x3max,y3min,y3max,z3min,z3max);
```

```
%%%%J6: row 7
g6=@(x,y,z,ii)(1-x).^(ii).*(1-y).^(d+db-k-1-ii).*f(x,y,z);
x6max=1;    y6max=1;        z6max=@(x,y)2-x/2-y/2;
x6min=eta;  y6min=@(x)x;    z6min=eta;
g6l=@(x,y,z)0;
for ii=0:1:d-k
    g6l=@(x,y,z)g6l(x,y,z)+g6(x,y,z,ii)*nchoosek(d+db-k-1,ii)...
        *nchoosek(d-2-ii,d-k-ii)*(-1)^(d-k-ii);
end
g6r=@(x,y,z)0;
for ii=0:1:db-k
    g6r=@(x,y,z)g6r(x,y,z)+g6(x,y,z,ii)*nchoosek(d+db-k-1,ii)...
        *nchoosek(db-2-ii,db-k-ii)*(-1)^(db-k-ii);
end
G6=@(x,y,z)g6l(x,y,z)+g6r(x,y,z);
J6=integral3(G6,x6min,x6max,y6min,y6max,z6min,z6max);

%%%%J7: row 8
J7=0;
    g7=@(x,y,z)(1-x-y).^(d+db-k-1).*f(x,y,z);
    x7max=1/2; y7max=@(x)1-x; z7max=@(x,y)2-x/2-y/2;
    x7min=eta; y7min=@(x)x; z7min=eta;
    J7=integral3(g7,x7min,x7max,y7min,y7max,z7min,z7max);
J7=J7*nchoosek(d+db-2*k,d-k);

%%%%J8: row 9-15
g8=@(x,y,z,ii)(1-(1-x).^(ii)-(1-y).^(ii)-(1-z).^(ii)+(1-x-y)....
   ^(ii)+(1-x-z).^(ii)+(1-y-z).^(ii)).*(1-x-y-z).^(d+db-k-1-...
   ii).*f(x,y,z);
x8max=1/3; y8max=@(x)1/2-x/2;   z8max=@(x,y)1-x-y;
x8min=eta; y8min=@(x)x;         z8min=@(x,y)y;
g8l=@(x,y,z)0;
for ii=0:1:d-k
    g8l=@(x,y,z)g8l(x,y,z)+g8(x,y,z,ii)*nchoosek(d+db-k-1,ii)...
        *nchoosek(d-2-ii,d-k-ii)*(-1)^(d-k-ii);
end
g8r=@(x,y,z)0;
for ii=0:1:db-k
    g8r=@(x,y,z)g8r(x,y,z)+g8(x,y,z,ii)*nchoosek(d+db-k-1,ii)...
        *nchoosek(db-2-ii,db-k-ii)*(-1)^(db-k-ii);
end
G8=@(x,y,z)g8l(x,y,z)+g8r(x,y,z);
J8=integral3(G8,x8min,x8max,y8min,y8max,z8min,z8max);

%%%%J9: row 16-18
g9=@(x,y,z,ii)((1-x).^(ii)-(1-x-y).^(ii)-(1-x-z).^(ii)).*(1-y...
   -z).^(d+db-k-1-ii).*f(x,y,z);
x9max=1/2; y9max=1/2;   z9max=@(x,y)1-y;
x9min=eta; y9min=@(x)x; z9min=@(x,y)y;
g9l=@(x,y,z)0;
for ii=0:1:d-k
    g9l=@(x,y,z)g9l(x,y,z)+g9(x,y,z,ii)*nchoosek(d+db-k-1,ii)...
        *nchoosek(d-2-ii,d-k-ii)*(-1)^(d-k-ii);
```

```
    end
    g9r=@(x,y,z)0;
    for ii=0:1:db-k
        g9r=@(x,y,z)g9r(x,y,z)+g9(x,y,z,ii)*nchoosek(d+db-k-1,ii)...
            *nchoosek(db-2-ii,db-k-ii)*(-1)^(db-k-ii);
    end
    G9=@(x,y,z)g9l(x,y,z)+g9r(x,y,z);
    J9=integral3(G9,x9min,x9max,y9min,y9max,z9min,z9max);

    %%%%J10: row 19-20
    g10=@(x,y,z,ii)((1-y).^(ii)-(1-x-y).^(ii)).*(1-x-z).^(d+db-k...
        -1-ii).*f(x,y,z);
    x10max=1/2; y10max=@(x)1-x; z10max=@(x,y)1-x;
    x10min=eta; y10min=@(x)x;   z10min=@(x,y)y;
    g10l=@(x,y,z)0;
    for ii=0:1:d-k
        g10l=@(x,y,z)g10l(x,y,z)+g10(x,y,z,ii)*nchoosek(d+db-k-1,...
            ii)*nchoosek(d-2-ii,d-k-ii)*(-1)^(d-k-ii);
    end
    g10r=@(x,y,z)0;
    for ii=0:1:db-k
        g10r=@(x,y,z)g10r(x,y,z)+g10(x,y,z,ii)*nchoosek(d+db-k-1,...
            ii)*nchoosek(db-2-ii,db-k-ii)*(-1)^(db-k-ii);
    end
    G10=@(x,y,z)g10l(x,y,z)+g10r(x,y,z);
    J10=integral3(G10,x10min,x10max,y10min,y10max,z10min,z10max);

    %%%%J11a: row 21a
    g11a=@(x,y,z,ii)((1-z).^(ii).*(1-x-y).^(d+db-k-1-ii)).*f(x,y,...
        z);
    x11max=1/2; y11max=@(x)1-x; z11max=@(x,y)x+y;
    x11min=eta; y11min=@(x)x; z11min=@(x,y)y;
    g11al=@(x,y,z)0;
    for ii=0:1:d-k
        g11al=@(x,y,z)g11al(x,y,z)+g11a(x,y,z,ii)*nchoosek(d+db-k...
            -1,ii)*nchoosek(d-2-ii,d-k-ii)*(-1)^(d-k-ii);
    end
    g11ar=@(x,y,z)0;
    for ii=0:1:db-k
        g11ar=@(x,y,z)g11ar(x,y,z)+g11a(x,y,z,ii)*nchoosek(d+db-k...
            -1,ii)*nchoosek(db-2-ii,db-k-ii)*(-1)^(db-k-ii);
    end
    G11a=@(x,y,z)g11al(x,y,z)+g11ar(x,y,z);
    J11a=integral3(G11a,x11min,x11max,y11min,y11max,z11min,z11max...
        );

    %%%%J11b: row 21b
    g11b=@(x,y,z,ii)((1-x-y).^(ii).*(1-z).^(d+db-k-1-ii)).*f(x,y,...
        z);
    x11max=1/2; y11max=@(x)1-x; z11max=1;
    x11min=eta; y11min=@(x)x; z11min=@(x,y)x+y;
    g11bl=@(x,y,z)0;
    for ii=0:1:d-k
        g11bl=@(x,y,z)g11bl(x,y,z)+g11b(x,y,z,ii)*nchoosek(d+db-k...
```

```
            -1,ii)*nchoosek(d-2-ii,d-k-ii)*(-1)^(d-k-ii);
    end
    g11br=@(x,y,z)0;
    for ii=0:1:db-k
        g11br=@(x,y,z)g11br(x,y,z)+g11b(x,y,z,ii)*nchoosek(d+db-k...
            -1,ii)*nchoosek(db-2-ii,db-k-ii)*(-1)^(db-k-ii);
    end
    G11b=@(x,y,z)g11bl(x,y,z)+g11br(x,y,z);
    J11b=integral3(G11b,x11min,x11max,y11min,y11max,z11min,z11max...
        );

    %%%%J14: row 22
    J14=0;
        g14=@(x,y,z)(1-x-y-z).^(d+db-k-1).*f(x,y,z);
        x14max=1/3; y14max=@(x)1/2-x/2; z14max=@(x,y)1-x-y;
        x14min=eta; y14min=@(x)x;        z14min=@(x,y)y;
        J14=integral3(g14,x14min,x14max,y14min,y14max,z14min,...
            z14max);
    J14=J14*nchoosek(d+db-2*k,d-k);

    Q=-J1+J2+J3+J6+J7-J8-J9-J10-J11a-J11b+J14;
    Q=Q*factorial(d-1)*factorial(db-1)/factorial(d+db-k-1);
```

A.4 Code for $v_{k,5}(m,n)$

```
function Q=Q5(k,d,db,eta)

    %%%%J1: row 2
    temp=@(x)0;
    for ii=0:1:d-k
        temp=@(x)temp(x)+nchoosek(d+db-k-1,ii)*nchoosek(d-2-ii,d-...
            k-ii)*(-1)^(d-k-ii)*(1-x).^(d+db-k-1-ii);
    end
    for ii=0:1:db-k
        temp=@(x)temp(x)+nchoosek(d+db-k-1,ii)*nchoosek(db-2-ii,...
            db-k-ii)*(-1)^(db-k-ii)*(1-x).^(d+db-k-1-ii);
    end
    f=@(x)temp(x).*integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t,...
       eta,1-x/4,@(y)y,@(y)4/3-x/3-y/3,@(y,z)z,@(y,z)2-x/2-y/2-z...
       /2);
    J1=integral(f,eta,1,'ArrayValued',true);

    %%%%J2: row 1,3
    f=@(x)(1-x).^(d+db-k-1).*integral3(@(y,z,t)1./(4-x-y-z-t)./x....
       /y./z./t,eta,1-x/4,@(y)y,@(y)4/3-x/3-y/3,@(y,z)z,@(y,z)2-...
       x/2-y/2-z/2);
    J2=integral(f,eta,1,'ArrayValued',true);
    f=@(x)integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t,x,1-x/4,@(...
       y)y,@(y)4/3-x/3-y/3,@(y,z)z,@(y,z)2-x/2-y/2-z/2);
    J2=J2+integral(f,eta,4/5,'ArrayValued',true);
    J2=J2*nchoosek(d+db-2*k,d-k);

    %%%%J3: row 4-6
    temp=@(x,y)0;
    for ii=0:1:d-k
        temp=@(x,y)temp(x,y)+(1-(1-x).^(ii)-(1-y).^(ii)).*(1-x-y)...
            .^(d+db-k-1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(d-2-ii...
            ,d-k-ii)*(-1)^(d-k-ii);
    end
    for ii=0:1:db-k
        temp=@(x,y)temp(x,y)+(1-(1-x).^(ii)-(1-y).^(ii)).*(1-x-y)...
            .^(d+db-k-1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(db-2-...
            ii,db-k-ii)*(-1)^(db-k-ii);
    end
    f=@(x)integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t.*temp(x,y)...
       ,x,1-x,eta,@(y)4/3-x/3-y/3,@(y,z)z,@(y,z)2-x/2-y/2-z/2);
    J3=integral(f,eta,1/2,'ArrayValued',true);

    %%%%J6: row 7
    temp=@(x,y)0;
    for ii=0:1:d-k
        temp=@(x,y)temp(x,y)+(1-x).^(ii).*(1-y).^(d+db-k-1-ii)*...
            nchoosek(d+db-k-1,ii)*nchoosek(d-2-ii,d-k-ii)*(-1)^(d...
            -k-ii);
    end
```

```
    for ii=0:1:db-k
        temp=@(x,y)temp(x,y)+(1-x).^(ii).*(1-y).^(d+db-k-1-ii)*...
            nchoosek(d+db-k-1,ii)*nchoosek(db-2-ii,db-k-ii)*(-1)...
            ^(db-k-ii);
    end
    f=@(x)integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t.*temp(x,y)...
        ,x,1,eta,@(y)4/3-x/3-y/3,@(y,z)z,@(y,z)2-x/2-y/2-z/2);
    J6=integral(f,eta,1,'ArrayValued',true);

    %%%%J7: row 8
    f=@(x)integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t.*(1-x-y)....
        ^(d+db-k-1),x,1-x,eta,@(y)4/3-x/3-y/3,@(y,z)z,@(y,z)2-x...
        /2-y/2-z/2);
    J7=integral(f,eta,1/2,'ArrayValued',true);
    J7=J7*nchoosek(d+db-2*k,d-k);

    %%%%J8: row 9-15
    temp=@(x,y,z)0;
    for ii=0:1:d-k
        temp=@(x,y,z)temp(x,y,z)+(1-(1-x).^(ii)-(1-y).^(ii)-(1-z)...
            .^(ii)+(1-x-y).^(ii)+(1-x-z).^(ii)+(1-y-z).^(ii))....
            *(1-x-y-z).^(d+db-k-1-ii)*nchoosek(d+db-k-1,ii)*...
            nchoosek(d-2-ii,d-k-ii)*(-1)^(d-k-ii);
    end
    for ii=0:1:db-k
        temp=@(x,y,z)temp(x,y,z)+(1-(1-x).^(ii)-(1-y).^(ii)-(1-z)...
            .^(ii)+(1-x-y).^(ii)+(1-x-z).^(ii)+(1-y-z).^(ii))....
            *(1-x-y-z).^(d+db-k-1-ii)*nchoosek(d+db-k-1,ii)*...
            nchoosek(db-2-ii,db-k-ii)*(-1)^(db-k-ii);
    end
    f=@(x)integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t.*temp(x,y,...
        z),x,1/2-x/2,@(y)y,@(y)1-x-y,eta,@(y,z)2-x/2-y/2-z/2);
    J8=integral(f,eta,1/3,'ArrayValued',true);

    %%%%J9: row 16-18
    temp=@(x,y,z)0;
    for ii=0:1:d-k
        temp=@(x,y,z)temp(x,y,z)+((1-x).^(ii)-(1-x-y).^(ii)-(1-x-...
            z).^(ii)).*(1-y-z).^(d+db-k-1-ii)*nchoosek(d+db-k-1,...
            ii)*nchoosek(d-2-ii,d-k-ii)*(-1)^(d-k-ii);
    end
    for ii=0:1:db-k
        temp=@(x,y,z)temp(x,y,z)+((1-x).^(ii)-(1-x-y).^(ii)-(1-x-...
            z).^(ii)).*(1-y-z).^(d+db-k-1-ii)*nchoosek(d+db-k-1,...
            ii)*nchoosek(db-2-ii,db-k-ii)*(-1)^(db-k-ii);
    end
    f=@(x)integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t.*temp(x,y,...
        z),x,1/2,@(y)y,@(y)1-y,eta,@(y,z)2-x/2-y/2-z/2);
    J9=integral(f,eta,1/2,'ArrayValued',true);

    %%%%J10: row 19-20
    temp=@(x,y,z)0;
    for ii=0:1:d-k
        temp=@(x,y,z)temp(x,y,z)+((1-y).^(ii)-(1-x-y).^(ii)).*(1-...
```

```
            x-z).^(d+db-k-1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(d...
            -2-ii,d-k-ii)*(-1)^(d-k-ii);
    end
    for ii=0:1:db-k
        temp=@(x,y,z)temp(x,y,z)+((1-y).^(ii)-(1-x-y).^(ii)).*(1-...
            x-z).^(d+db-k-1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(db...
            -2-ii,db-k-ii)*(-1)^(db-k-ii);
    end
    f=@(x)integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t.*temp(x,y,...
        z),x,1-x,@(y)y,1-x,eta,@(y,z)2-x/2-y/2-z/2);
    J10=integral(f,eta,1/2,'ArrayValued',true);

    %%%%J11: row 21
    temp=@(x,y,z)0;
    for ii=0:1:d-k
        temp=@(x,y,z)temp(x,y,z)+(1-z).^(ii).*(1-x-y).^(d+db-k-1-...
            ii)*nchoosek(d+db-k-1,ii)*nchoosek(d-2-ii,d-k-ii)...
            *(-1)^(d-k-ii);
    end
    for ii=0:1:db-k
        temp=@(x,y,z)temp(x,y,z)+(1-z).^(ii).*(1-x-y).^(d+db-k-1-...
            ii)*nchoosek(d+db-k-1,ii)*nchoosek(db-2-ii,db-k-ii)...
            *(-1)^(db-k-ii);
    end
    f=@(x)integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t.*temp(x,y,...
        z),x,1-x,@(y)y,@(y)x+y,eta,@(y,z)2-x/2-y/2-z/2);
    J11=integral(f,eta,1/2,'ArrayValued',true);
    temp=@(x,y,z)0;
    for ii=0:1:d-k
        temp=@(x,y,z)temp(x,y,z)+(1-x-y).^(ii).*(1-z).^(d+db-k-1-...
            ii)*nchoosek(d+db-k-1,ii)*nchoosek(d-2-ii,d-k-ii)...
            *(-1)^(d-k-ii);
    end
    for ii=0:1:db-k
        temp=@(x,y,z)temp(x,y,z)+(1-x-y).^(ii).*(1-z).^(d+db-k-1-...
            ii)*nchoosek(d+db-k-1,ii)*nchoosek(db-2-ii,db-k-ii)...
            *(-1)^(db-k-ii);
    end
    f=@(x)integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t.*temp(x,y,...
        z),x,1-x,@(y)x+y,1,eta,@(y,z)2-x/2-y/2-z/2);
    J11=J11+integral(f,eta,1/2,'ArrayValued',true);

    %%%%J14: row 22
    f=@(x)integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t.*(1-x-y-z)...
        .^(d+db-k-1),x,1/2-x/2,@(y)y,@(y)1-x-y,eta,@(y,z)2-x/2-y...
        /2-z/2);
    J14=integral(f,eta,1/3,'ArrayValued',true);
    J14=J14*nchoosek(d+db-2*k,d-k);

    %%%%JI: row 23-37
    temp=@(x,y,z,t)0;
    for ii=0:1:d-k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1-(1-x).^(ii)-(1-y).^(ii)...
            -(1-z).^(ii)-(1-t).^(ii)+(1-x-y).^(ii)+(1-x-z).^(ii)...
```

```
            +(1-x-t).^(ii)+(1-y-z).^(ii)+(1-y-t).^(ii)+(1-z-t).^(...
            ii)-(1-x-y-z).^(ii)-(1-x-y-t).^(ii)-(1-x-z-t).^(ii)...
            -(1-y-z-t).^(ii)).*(1-x-y-z-t).^(d+db-k-1-ii)*...
            nchoosek(d+db-k-1,ii)*nchoosek(d-2-ii,d-k-ii)*(-1)^(d...
            -k-ii);
    end
    for ii=0:1:db-k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1-(1-x).^(ii)-(1-y).^(ii)...
            -(1-z).^(ii)-(1-t).^(ii)+(1-x-y).^(ii)+(1-x-z).^(ii)...
            +(1-x-t).^(ii)+(1-y-z).^(ii)+(1-y-t).^(ii)+(1-z-t).^(...
            ii)-(1-x-y-z).^(ii)-(1-x-y-t).^(ii)-(1-x-z-t).^(ii)...
            -(1-y-z-t).^(ii)).*(1-x-y-z-t).^(d+db-k-1-ii)*...
            nchoosek(d+db-k-1,ii)*nchoosek(db-2-ii,db-k-ii)*(-1)...
            ^(db-k-ii);
    end
    f=@(x)integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t.*temp(x,y,...
        z,t),x,1/3-x/3,@(y)y,@(y)1/2-x/2-y/2,@(y,z)z,@(y,z)1-x-y-...
        z);
    JI=integral(f,eta,1/4,'ArrayValued',true);

    %%%%JII: row 38-44
    temp=@(x,y,z,t)0;
    for ii=0:1:d-k
        temp=@(x,y,z,t)temp(x,y,z,t)+((1-x).^(ii)-(1-x-y).^(ii)...
            -(1-x-z).^(ii)-(1-x-t).^(ii)+(1-x-y-z).^(ii)+(1-x-y-t...
            ).^(ii)+(1-x-z-t).^(ii)).*(1-y-z-t).^(d+db-k-1-ii)*...
            nchoosek(d+db-k-1,ii)*nchoosek(d-2-ii,d-k-ii)*(-1)^(d...
            -k-ii);
    end
    for ii=0:1:db-k
        temp=@(x,y,z,t)temp(x,y,z,t)+((1-x).^(ii)-(1-x-y).^(ii)...
            -(1-x-z).^(ii)-(1-x-t).^(ii)+(1-x-y-z).^(ii)+(1-x-y-t...
            ).^(ii)+(1-x-z-t).^(ii)).*(1-y-z-t).^(d+db-k-1-ii)*...
            nchoosek(d+db-k-1,ii)*nchoosek(db-2-ii,db-k-ii)*(-1)...
            ^(db-k-ii);
    end
    f=@(x)integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t.*temp(x,y,...
        z,t),x,1/3,@(y)y,@(y)1/2-y/2,@(y,z)z,@(y,z)1-y-z);
    JII=integral(f,eta,1/3,'ArrayValued',true);

    %%%%JIII: row 45-50
    temp=@(x,y,z,t)0;
    for ii=0:1:d-k
        temp=@(x,y,z,t)temp(x,y,z,t)+((1-y).^(ii)-(1-x-y).^(ii)...
            -(1-y-z).^(ii)-(1-y-t).^(ii)+(1-x-y-z).^(ii)+(1-x-y-t...
            ).^(ii)).*(1-x-z-t).^(d+db-k-1-ii)*nchoosek(d+db-k-1,...
            ii)*nchoosek(d-2-ii,d-k-ii)*(-1)^(d-k-ii);
    end
    for ii=0:1:db-k
        temp=@(x,y,z,t)temp(x,y,z,t)+((1-y).^(ii)-(1-x-y).^(ii)...
            -(1-y-z).^(ii)-(1-y-t).^(ii)+(1-x-y-z).^(ii)+(1-x-y-t...
            ).^(ii)).*(1-x-z-t).^(d+db-k-1-ii)*nchoosek(d+db-k-1,...
            ii)*nchoosek(db-2-ii,db-k-ii)*(-1)^(db-k-ii);
    end
```

```
f=@(x)integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t.*temp(x,y,...
    z,t),x,1/2-x/2,@(y)y,1/2-x/2,@(y,z)z,@(y,z)1-x-z);
JIII=integral(f,eta,1/3,'ArrayValued',true);

%%%%JIV: row 51-53,55
temp=@(x,y,z,t)0;
for ii=2:1:d-k %for loop starts from 2, since the integrand ...
    is zero when ii=0,1
    temp=@(x,y,z,t)temp(x,y,z,t)+((1-z).^(ii)-(1-x-z).^(ii)...
        -(1-y-z).^(ii)+(1-x-y-z).^(ii)).*(1-x-y-t).^(d+db-k...
        -1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(d-2-ii,d-k-ii)...
        *(-1)^(d-k-ii);
end
for ii=2:1:db-k %for loop starts from 2, since the integrand ...
    is zero when ii=0,1
    temp=@(x,y,z,t)temp(x,y,z,t)+((1-z).^(ii)-(1-x-z).^(ii)...
        -(1-y-z).^(ii)+(1-x-y-z).^(ii)).*(1-x-y-t).^(d+db-k...
        -1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(db-2-ii,db-k-ii...
        )*(-1)^(db-k-ii);
end
f=@(x)integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t.*temp(x,y,...
    z,t),x,1/2-x/2,@(y)y,@(y)1-x-y,@(y,z)z,@(y,z)1-x-y);
JIV=integral(f,eta,1/3,'ArrayValued',true);

%%%%JV: row 54
temp=@(x,y,z,t)0;
for ii=0:1:d-k
    temp=@(x,y,z,t)temp(x,y,z,t)+(1-z-t).^(ii).*(1-x-y-t).^(d...
        +db-k-1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(d-2-ii,d-k...
        -ii)*(-1)^(d-k-ii);
end
for ii=0:1:db-k
    temp=@(x,y,z,t)temp(x,y,z,t)+(1-z-t).^(ii).*(1-x-y-t).^(d...
        +db-k-1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(db-2-ii,db...
        -k-ii)*(-1)^(db-k-ii);
end
f=@(x)integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t.*temp(x,y,...
    z,t),x,1/2-x/2,@(y)y,@(y)min(x+y,1-x-y),@(y,z)z,@(y,z)1-x...
    -y);
JV=integral(f,eta,1/3,'ArrayValued',true);
temp=@(x,y,z,t)0;
for ii=0:1:d-k
    temp=@(x,y,z,t)temp(x,y,z,t)+(1-x-y-t).^(ii).*(1-z-t).^(d...
        +db-k-1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(d-2-ii,d-k...
        -ii)*(-1)^(d-k-ii);
end
for ii=0:1:db-k
    temp=@(x,y,z,t)temp(x,y,z,t)+(1-x-y-t).^(ii).*(1-z-t).^(d...
        +db-k-1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(db-2-ii,db...
        -k-ii)*(-1)^(db-k-ii);
end
f=@(x)integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t.*temp(x,y,...
    z,t),x,1/2-x,@(y)x+y,1/2,@(y,z)z,@(y,z)1-z);
JV=JV+integral(f,eta,1/4,'ArrayValued',true);
```

```
    %%%%JVI: row 56
    temp=@(x,y,z,t)0;
    for ii=0:1:d-k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1-t).^(ii).*(1-x-y-z).^(d+...
           db-k-1-ii)*nchoose k(d+db-k-1,ii)*nchoosek(d-2-ii,d-k-...
           ii)*(-1)^(d-k-ii);
    end
    for ii=0:1:db-k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1-t).^(ii).*(1-x-y-z).^(d+...
           db-k-1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(db-2-ii,db-...
           k-ii)*(-1)^(db-k-ii);
    end
    f=@(x)integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t.*temp(x,y,...
        z,t),x,1/2-x/2,@(y)y,@(y)1-x-y,@(y,z)z,@(y,z)x+y+z);
    JVI=integral(f,eta,1/3,'ArrayValued',true);
    temp=@(x,y,z,t)0;
    for ii=0:1:d-k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1-x-y-z).^(ii).*(1-t).^(d+...
           db-k-1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(d-2-ii,d-k-...
           ii)*(-1)^(d-k-ii);
    end
    for ii=0:1:db-k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1-x-y-z).^(ii).*(1-t).^(d+...
           db-k-1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(db-2-ii,db-...
           k-ii)*(-1)^(db-k-ii);
    end
    f=@(x)integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t.*temp(x,y,...
        z,t),x,1/2-x/2,@(y)y,@(y)1-x-y,@(y,z)x+y+z,1);
    JVI=JVI+integral(f,eta,1/3,'ArrayValued',true);

    %%%%JVII: row 57
    temp=@(x,y,z,t)0;
    for ii=0:1:d-k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1-x-t).^(ii).*(1-x-y-z).^(d...
           +db-k-1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(d-2-ii,d-k...
           -ii)*(-1)^(d-k-ii);
    end
    for ii=0:1:db-k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1-x-t).^(ii).*(1-x-y-z).^(d...
           +db-k-1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(db-2-ii,db...
           -k-ii)*(-1)^(db-k-ii);
    end
    f=@(x)integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t.*temp(x,y,...
        z,t),x,1/2-x/2,@(y)y,@(y)1-x-y,@(y,z)z,@(y,z)y+z);
    JVII=integral(f,eta,1/3,'ArrayValued',true);
    temp=@(x,y,z,t)0;
    for ii=0:1:d-k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1-x-y-z).^(ii).*(1-x-t).^(d...
           +db-k-1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(d-2-ii,d-k...
           -ii)*(-1)^(d-k-ii);
    end
    for ii=0:1:db-k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1-x-y-z).^(ii).*(1-x-t).^(d...
```

```
            +db−k−1−ii)*nchoosek(d+db−k−1,ii)*nchoosek(db−2−ii,db...
            −k−ii)*(−1)^(db−k−ii);
    end
    f=@(x)integral3(@(y,z,t)1./(4−x−y−z−t)./x./y./z./t.*temp(x,y,...
        z,t),x,1/2−x/2,@(y)y,@(y)1−x−y,@(y,z)y+z,1−x);
    JVII=JVII+integral(f,eta,1/3,'ArrayValued',true);

    %%%%JVIII: row 58
    temp=@(x,y,z,t)0;
    for ii=0:1:d−k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1−y−t).^(ii).*(1−x−y−z).^(d...
            +db−k−1−ii)*nchoosek(d+db−k−1,ii)*nchoosek(d−2−ii,d−k...
            −ii)*(−1)^(d−k−ii);
    end
    for ii=0:1:db−k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1−y−t).^(ii).*(1−x−y−z).^(d...
            +db−k−1−ii)*nchoosek(d+db−k−1,ii)*nchoosek(db−2−ii,db...
            −k−ii)*(−1)^(db−k−ii);
    end
    f=@(x)integral3(@(y,z,t)1./(4−x−y−z−t)./x./y./z./t.*temp(x,y,...
        z,t),x,1/2−x/2,@(y)y,@(y)1−x−y,@(y,z)z,@(y,z)x+z);
    JVIII=integral(f,eta,1/3,'ArrayValued',true);
    temp=@(x,y,z,t)0;
    for ii=0:1:d−k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1−x−y−z).^(ii).*(1−y−t).^(d...
            +db−k−1−ii)*nchoosek(d+db−k−1,ii)*nchoosek(d−2−ii,d−k...
            −ii)*(−1)^(d−k−ii);
    end
    for ii=0:1:db−k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1−x−y−z).^(ii).*(1−y−t).^(d...
            +db−k−1−ii)*nchoosek(d+db−k−1,ii)*nchoosek(db−2−ii,db...
            −k−ii)*(−1)^(db−k−ii);
    end
    f=@(x)integral3(@(y,z,t)1./(4−x−y−z−t)./x./y./z./t.*temp(x,y,...
        z,t),x,1/2−x/2,@(y)y,@(y)1−x−y,@(y,z)x+z,@(y,z)1−y);
    JVIII=JVIII+integral(f,eta,1/3,'ArrayValued',true);

    %%%%JIX: row 59
    temp=@(x,y,z,t)0;
    for ii=0:1:d−k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1−z−t).^(ii).*(1−x−y−z).^(d...
            +db−k−1−ii)*nchoosek(d+db−k−1,ii)*nchoosek(d−2−ii,d−k...
            −ii)*(−1)^(d−k−ii);
    end
    for ii=0:1:db−k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1−z−t).^(ii).*(1−x−y−z).^(d...
            +db−k−1−ii)*nchoosek(d+db−k−1,ii)*nchoosek(db−2−ii,db...
            −k−ii)*(−1)^(db−k−ii);
    end
    f=@(x)integral3(@(y,z,t)1./(4−x−y−z−t)./x./y./z./t.*temp(x,y,...
        z,t),x,1/2−x/2,@(y)y,@(y)min(x+y,1−x−y),@(y,z)z,@(y,z)x+y...
        );
    JIX=integral(f,eta,1/3,'ArrayValued',true);
    temp=@(x,y,z,t)0;
```

```
    for ii=0:1:d-k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1-x-y-z).^(ii).*(1-z-t).^(d...
            +db-k-1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(d-2-ii,d-k...
            -ii)*(-1)^(d-k-ii);
    end
    for ii=0:1:db-k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1-x-y-z).^(ii).*(1-z-t).^(d...
            +db-k-1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(db-2-ii,db...
            -k-ii)*(-1)^(db-k-ii);
    end
    f=@(x)integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t.*temp(x,y,...
        z,t),x,1/2-x/2,@(y)y,@(y)min(1-x-y,0.5),@(y,z)max(x+y,z),...
        @(y,z)1-z);
    JIX=JIX+integral(f,eta,1/3,'ArrayValued',true);

    %%%%JX: row 60
    temp=@(x,y,z,t)0;
    for ii=0:1:d-k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1-x-y).^(ii).*(1-z-t).^(d+...
            db-k-1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(d-2-ii,d-k-...
            ii)*(-1)^(d-k-ii);
    end
    for ii=0:1:db-k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1-x-y).^(ii).*(1-z-t).^(d+...
            db-k-1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(db-2-ii,db-...
            k-ii)*(-1)^(db-k-ii);
    end
    f=@(x)integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t.*temp(x,y,...
        z,t),x,1/2,@(y)y,1/2,@(y,z)z,@(y,z)1-z);
    JX=integral(f,eta,1/2,'ArrayValued',true);

    %%%%JXI: row 61
    temp=@(x,y,z,t)0;
    for ii=0:1:d-k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1-x-z).^(ii).*(1-y-t).^(d+...
            db-k-1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(d-2-ii,d-k-...
            ii)*(-1)^(d-k-ii);
    end
    for ii=0:1:db-k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1-x-z).^(ii).*(1-y-t).^(d+...
            db-k-1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(db-2-ii,db-...
            k-ii)*(-1)^(db-k-ii);
    end
    f=@(x)integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t.*temp(x,y,...
        z,t),x,1/2,@(y)y,@(y)1-y,@(y,z)z,@(y,z)1-y);
    JXI=integral(f,eta,1/2,'ArrayValued',true);

    %%%%JXII: row 62
    temp=@(x,y,z,t)0;
    for ii=0:1:d-k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1-y-z).^(ii).*(1-x-t).^(d+...
            db-k-1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(d-2-ii,d-k-...
            ii)*(-1)^(d-k-ii);
    end
```

```
    for ii=0:1:db-k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1-y-z).^(ii).*(1-x-t).^(d+...
            db-k-1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(db-2-ii,db-...
            k-ii)*(-1)^(db-k-ii);
    end
    f=@(x)integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t.*temp(x,y,...
        z,t),x,1/2,@(y)y,@(y)1-y,@(y,z)y+z-x,1-x);
    JXII=integral(f,eta,1/2,'ArrayValued',true);
    temp=@(x,y,z,t)0;
    for ii=0:1:d-k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1-x-t).^(ii).*(1-y-z).^(d+...
            db-k-1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(d-2-ii,d-k-...
            ii)*(-1)^(d-k-ii);
    end
    for ii=0:1:db-k
        temp=@(x,y,z,t)temp(x,y,z,t)+(1-x-t).^(ii).*(1-y-z).^(d+...
            db-k-1-ii)*nchoosek(d+db-k-1,ii)*nchoosek(db-2-ii,db-...
            k-ii)*(-1)^(db-k-ii);
    end
    f=@(x)integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t.*temp(x,y,...
        z,t),x,1/2,@(y)y,@(y)1-y,@(y,z)z,@(y,z)y+z-x);
    JXII=JXII+integral(f,eta,1/2,'ArrayValued',true);

    %%%%JXIII: row 63
    f=@(x)integral3(@(y,z,t)1./(4-x-y-z-t)./x./y./z./t.*(1-x-y-z-...
        t).^(d+db-k-1),x,1/3-x/3,@(y)y,@(y)1/2-x/2-y/2,@(y,z)z,@(...
        y,z)1-x-y-z);
    JXIII=integral(f,eta,1/4,'ArrayValued',true);
    JXIII=JXIII*nchoosek(d+db-2*k,d-k);

    Q=-J1+J2+J3+J6+J7-J8-J9-J10-J11+J14+JI+JII+JIII+JIV-JV+JVI-...
        JVII-JVIII-JIX+JX+JXI+JXII+JXIII;
    Q=Q*factorial(d-1)*factorial(db-1)/factorial(d+db-k-1);
```

Appendix B

Record of $V_{k,j}(m)$

(a)

$$V_{2,0} = \begin{pmatrix} 2.0000 & 2.0000 \\ 2.0000 & 3.0000 \end{pmatrix} \quad , \quad V_{2,1} = \begin{pmatrix} 0.3333 & 0.3750 \\ 0.3750 & 0.4500 \end{pmatrix},$$
$$V_{2,2} = \begin{pmatrix} 0.6947 & 0.8163 \\ 0.8163 & 1.0796 \end{pmatrix} \quad , \quad V_{2,3} = \begin{pmatrix} 0.5015 & 0.6034 \\ 0.6034 & 0.9529 \end{pmatrix}.$$

(b)

$$V_{3,0} = \begin{pmatrix} 6.0000 & 6.0000 \\ 6.0000 & 9.6000 \end{pmatrix} \quad , \quad V_{3,1} = \begin{pmatrix} 0.7500 & 0.9000 \\ 0.9000 & 1.2000 \end{pmatrix},$$
$$V_{3,2} = \begin{pmatrix} 1.7720 & 2.1592 \\ 2.1592 & 3.0977 \end{pmatrix} \quad , \quad V_{3,3} = \begin{pmatrix} 1.5948 & 1.9058 \\ 1.9058 & 3.0850 \end{pmatrix}.$$

(c/d)

$$V_{4,0} = \begin{pmatrix} 24.0000 & 24.0000 \\ 24.0000 & 40.0000 \end{pmatrix} \quad , \quad V_{4,1} = \begin{pmatrix} 2.4000 & 3.0000 \\ 3.0000 & 4.2857 \end{pmatrix},$$
$$V_{4,2} = \begin{pmatrix} 6.1954 & 7.7442 \\ 7.7442 & 11.7249 \end{pmatrix} \quad , \quad V_{4,3} = \begin{pmatrix} 6.4011 & 7.7124 \\ 7.7124 & 12.7732 \end{pmatrix},$$
$$V_{4,4} = \begin{pmatrix} 3.8734 & 3.9399 \\ 3.9399 & 7.4547 \end{pmatrix} \quad , \quad V_{4,5} = \begin{pmatrix} 1.9497 & 1.1896 \\ 1.1896 & 2.6990 \end{pmatrix}.$$

(e)

$$V_{5,0} = \begin{pmatrix} 120.0000 & 120.0000 & 120.0000 \\ 120.0000 & 205.7143 & 270.0000 \\ 120.0000 & 270.0000 & 420.0000 \end{pmatrix}, \quad V_{5,1} = \begin{pmatrix} 10.0000 & 12.8571 & 15.0000 \\ 12.8571 & 19.2857 & 25.0000 \\ 15.0000 & 25.0000 & 35.0000 \end{pmatrix},$$

$$V_{5,2} = \begin{pmatrix} 27.6124 & 35.1746 & 40.3540 \\ 35.1746 & 55.3437 & 72.9153 \\ 40.3540 & 72.9153 & 105.7971 \end{pmatrix}, \quad V_{5,3} = \begin{pmatrix} 31.4693 & 38.3197 & 41.8529 \\ 38.3197 & 64.6991 & 86.6771 \\ 41.8529 & 86.6771 & 132.3505 \end{pmatrix},$$

$$V_{5,4} = \begin{pmatrix} 21.3554 & 22.3641 & 21.0342 \\ 22.3641 & 41.6774 & 56.1494 \\ 21.0342 & 56.1494 & 91.9789 \end{pmatrix}, \quad V_{5,5} = \begin{pmatrix} 11.5004 & 8.0970 & 4.3296 \\ 8.0970 & 17.1317 & 22.2077 \\ 4.3296 & 22.2077 & 40.0247 \end{pmatrix}.$$

Bibliography

[1] Apostol, T. M., *Mathematical Analysis, 2nd Edition*, Addison-Wesley Pub. Co., 1974.

[2] Bombieri, E., Friedlander, J. B., and Iwaniec, H., *Primes in arithmetic progressions to large moduli*, Acta Math., **156**(1986), 203-251.

[3] Conway, J. B., *Functions of One Complex Variable, 2nd Edition*, Graduate Texts in Math. 11, Springer, 1978.

[4] Davenport, H., *Multiplicative Number Theory, 3rd Edition*, Graduate Texts in Math. 74, Springer, 2000.

[5] Goldston, D. A., Graham, S. W., Pintz, J., and Yıldırım, C. Y., *Small gaps between products of two primes*, Proc. Lond. Math. Soc. **3**(2009), 98, 741-774.

[6] Goldston, D. A., Graham, S. W., Pintz, J., and Yıldırım, C. Y., *Small gaps between almost primes, the parity problem, and some conjectures of Erdös on consecutive integers*, Int. Math. Res. Notices (2011), no.7, 1439-1450.

[7] Goldston, D. A., Motohashi, Y., Pintz, J., and Yıldırım, C. Y., *Small gaps between primes exist*, Proc. Japan Acad. Ser. A Math. Sci. **82**(2006), 61-65.

[8] Halberstam, H., and Richert H.-E., *Sieve methods*, Academic Press, London, New York, 1974.

[9] Hardy, G. H., and Littlewood, J. E., *Some problems of 'Partitio Numerorum'; III: On the expression of a number as a sum of primes*, Acta Math. **44**(1922), 1-70.

[10] Heath-Brown, D. R., *Almost prime k-tuples*, Mathematatika **44**(1997), no.2, 245-266.

[11] Ho, K. H., and Tsang, K. M., *On almost prime k-tuples*, J. Number Theory **120**(2006), no.1, 33-46.

[12] Maynard, J., *Almost-prime k-tuples*, Pre-print.

[13] Motohashi, Y., *An induction principle for the generalization of Bombieri's prime number theorem*, Proc. Japan Acad. **52**,(1976), no.6, 273-275.

[14] Nathanson, M. B., *Additive Number Theory: The Classical Bases*, Graduate Texts in Math. 164, Springer, 1996.

[15] Selberg, A., *Contributions to the theory of the Riemann zeta-function* Arch. Math. Naturvid. **48**(1946), no. 5, 89-155; Collected papers Vol.I, Springer, Berlin (1989), 214-280.

[16] Selberg, A., *Lectures on sieves*, Collected Papers Vol.II, Springer-Verlag (1991), 65-247.

[17] Soundararajan, K., *Small gaps between prime numbers: the work of Goldston-Pintz-Yıldırım*, Bull. Amer. Math. Soc. **44**(2007), no.1, 1-18.

[18] Titchmarsh, E. C., *The Theory of the Riemann zeta-function, 2nd Edition*, Oxford Univ. Press, 1986.

[19] Thorne, F., *Bounded gaps between products of primes with applications to ideal class groups and elliptic curves*, Int. Math. Res. Notices (2008), rnm156, 41 pages.

[20] Widder, D. V., *Advanced calculus, 2nd Edition*, Prentice-Hall of India Pvt. Ltd., 1964.

[21] Zhang, Y. T., *Bounded gaps between primes*, Ann. of Math., to appear.