# Numeration Systems with Finite Type Condition

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Abstract of thesis entitled

# NUMERATION SYSTEMS WITH FINITE TYPE CONDITION

Submitted by

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Let $q > 1$ and $\mathsf{D} = \{\partial_0, \dots, \partial_m\}$ be a set of real numbers such that $0 < \partial_{i+1} - \partial_i \leq (\partial_m - \partial_0)/(q-1)$ for all $0 \leq i \leq m-1$. The pair $(q, \mathsf{D})$ can be treated as an iterated function system or a numeration system. This thesis investigates when $\Phi = (q, \mathsf{D})$ satisfies the finite type condition (FTC), that is, when the set

$$\left\{ \sum_{i=0}^{n} q^i (s_i - t_i) : \ n \geq 0, \ s_i, t_i \in \mathsf{D} \right\}$$

has no accumulation point. It is proved that

I. If $1 \leq m \leq 3$ and $\Phi$ satisfies the FTC, then $q$ is a Pisot–Vijayaraghavan number (PV number).

II. If $m = 4$ and $\Phi$ satisfies the FTC, then every algebraic conjugate $A$ of $q$ with $|A| \geq 1$ is not a real number.

III. For each $m \geq 4$, there exists $\Phi$ satisfying the FTC, and the associated $q$ is not a PV number.

# 記數系統和有限類條件


## 摘要

設 $q>1$，$\mathsf{D}=\{\partial_0,\ldots,\partial_m\}$ 為一組實數，其中對於所有 $0\le i\le m-1$，皆有 $0<\partial_{i+1}-\partial_i\le(\partial_m-\partial_0)/(q-1)$ 成立。這樣一對 $(q,\mathsf{D})$ 可當成一個迭代函數系統或一個記數系統。本論文研究 $\Phi=(q,\mathsf{D})$ 如何才滿足有限類條件 (Finite Type Condition，FTC)，即集合

$$\left\{\sum_{i=0}^{n} q^i(s_i-t_i):\ n\ge 0,\ s_i,t_i\in\mathsf{D}\right\}$$

如何才不會有聚點。本論文證明

I. 如果 $1\le m\le 3$，且 $\Phi$ 滿足 FTC，則 $q$ 是一個 Pisot–Vijayaraghavan 數 (PV 數)。

II. 如果 $m=4$，且 $\Phi$ 滿足 FTC，則對於所有 $q$ 的共軛代數數 $A$，如果 $|A|\ge 1$，那麼 $A$ 便不是實數。

III. 對於所有 $m\ge 4$，都存在一個滿足 FTC 的 $\Phi$，且當中的 $q$ 並不是 PV 數。

# NUMERATION SYSTEMS WITH FINITE TYPE CONDITION

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# Declaration

I declare that this thesis represents my own work, except where due acknowledgement is made, and that it has not been previously included in a thesis, dissertation or report submitted to this University or to any other institution for a degree, diploma or other qualifications.

CHAN, Ching Yin

## Acknowledgements

在二零二零年，我得到了這部作品中的大部分結果。不知是天意還是我的意志，讓我在這個時代，完成了這件事。在此感謝研究生導師丰德軍教授，令我得到一個不錯的結局。

我要在作品的開頭記下一段話。這件事能夠完成，這部作品有這個模樣，就是因為這些年華和時光。

此頁告成。下頁是新的一頁。

# Contents

如果這份力氣還讓我有足夠多的時間完成我的作品，

那麼，至少我誤不了在作品中首先要描繪那些人
（哪怕把他們寫得像怪物），

寫出他們佔有那麼巨大的地盤，
相比之下在空間中為他們保留的位置是那麼狹隘，
相反，他們卻佔有一個無限度延續的位置，

因為他們像潛入似水年華的巨人，
同時觸及間隔甚遠的幾個時代，
時代與時代之間安置了那麼多的日子

——在時間之中。

馬塞爾・普魯斯特，追憶似水年華：重現的時光
(徐和瑾　周國強譯)

Aussi, si elle m'était laissée assez longtemps pour accomplir mon œuvre, ne manquerais-je pas d'abord d'y décrire les hommes, cela dût-il les faire ressembler à des êtres monstrueux, comme occupant une place si considérable, à côté de celle si restreinte qui leur est réservée dans l'espace, une place au contraire prolongée sans mesure puisqu'ils touchent simultanément, comme des géants plongés dans les années à des époques, vécues par eux si distantes, entre lesquelles tant de jours sont venus se placer — dans le Temps.

Marcel Proust, *À la recherche du temps perdu: Le Temps retrouvé*

But at least, if strength were granted me for long enough to accomplish my work, I should not fail, even if the effect were to make them resemble monsters, to describe men first and foremost as occupying so considerable a place, compared with the restricted place which is reserved for them in space, a place on the contrary prolonged past measure, for simultaneously, like giants plunged into the years, they touch the distant epochs through which they have lived, between which so many days have come to range themselves — in Time.

Marcel Proust, *In Search of Lost Time: Time Regained*
(Translated by Andreas Mayor and Terence Kilmartin)

# Chapter 1

# Background

## 1.1 Prolegomena

In the field of fractal geometry, a central theme is to study the dimensions of fractals [18]. Roughly speaking, fractal dimensions are non-negative real numbers generalizing the notion of dimensions of familiar geometric objects (e.g. straight lines, surfaces) to fractal sets, giving quantitative descriptions of how the space is occupied [18]. Usually, the estimation of dimensions is not an easy task. Noting that a large class of interesting fractals, including the middle third Cantor set, the von Koch curve, and the Sierpinski gasket, can be constructed by using iterated function systems (IFS), mathematicians find that if an IFS satisfies certain separation conditions, then an understanding of the associated fractal and its dimensions follows. An example of such conditions is the finite type condition (FTC). If an IFS satisfies the FTC, then the associated fractal will only repeat finitely many overlapping patterns in all scales. Therefore, we are able to reach an understanding of the fractal by a finite number of steps.

The FTC also plays a role in the study of numeration systems (also called number systems) in number theory and computer science. We are used to representing real numbers by decimal expansions, which use $\beta = 10$ as the base and $\mathsf{D} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ as the set of digits. If we call $(\beta, \mathsf{D})$ a numeration system, then as a generalization, different $\beta$ and $\mathsf{D}$ may also be used. The new system obtained may or may not possess the same properties as the ordinary decimal system does, and the resulting expansions are called $\beta$-expansions in the literature. Noting that the decimal expansion of a real number can be obtained by an iterative algorithm, we see that each numeration system corresponds to an IFS on $\mathbb{R}$. It follows that if an IFS on $\mathbb{R}$ satisfies the FTC , then we immediately have an understanding of the structure of the associated numeration system.

In view of the above discussion, it is natural to ask: What are the necessary and sufficient conditions for an IFS to satisfy the FTC? Are there any simple criteria or algorithms for verifying if an IFS satisfies the FTC? Up to now, the existing literature only provides partial answers to these questions, and it is observed that Pisot–Vijayaraghavan numbers (PV numbers for short) may play a crucial role in this problem (a PV number is a real algebraic integer which is strictly greater than 1, while all of its algebraic conjugates have absolute values strictly less than 1). Although there are examples showing that the involvement of PV numbers is not necessary for the FTC (e.g. [40, Example 5.5]), all of them have holes in their "attractors". On the other hand, it was found that for some class of IFS on $\mathbb{R}$ without hole in their attractors, the necessary and sufficient condition is that the parameters involved must be PV numbers [1, 22]. Does the same hold for a wider class of IFS on $\mathbb{R}$? If not, then how do we construct counterexamples and how do they look like? To date, there are only a few literature on this problem, and the results obtained still do not provide a complete picture. In view of the relationship between this problem and the study of fractal geometry, number theory and computer science, a further investigation of the issue should be beneficial.

## 1.2 Purpose of the research

We consider a class of homogeneous iterated function systems (IFS) on $\mathbb{R}$ having the form $\Theta \stackrel{\text{def}}{=} \{\rho x + c_i\}_{i=0}^m$ with $0 < \rho < 1$, $c_0 < c_1 < \cdots < c_m$, and the attractor being a closed interval $[L, R]$, so that

$$[L, R] = \bigcup_{i=0}^{m} [\rho L + c_i,\ \rho R + c_i].$$

This implies

(i) $\rho L + c_{i+1} \leq \rho R + c_i$, whence $c_{i+1} - c_i \leq \rho(R - L)$;

(ii) $c_0 = (1-\rho)L$ and $c_m = (1-\rho)R$.

Let $q = q(\Theta) \stackrel{\text{def}}{=} 1/\rho$, $\mathsf{D}_\Theta \stackrel{\text{def}}{=} \{qc_i\}_{i=0}^m$, $(\mathsf{D}_\Theta - \mathsf{D}_\Theta) \stackrel{\text{def}}{=} \{\delta_1 - \delta_2 : \delta_1, \delta_2 \in \mathsf{D}_\Theta\}$, and

$$Y_\Theta \stackrel{\text{def}}{=} \left\{ \sum_{i=0}^{n} q^i s_i : n \geq 0,\ s_i \in (\mathsf{D}_\Theta - \mathsf{D}_\Theta) \right\}.$$

In view of numeration system, we may call $q$ a base, $\mathsf{D}_\Theta$ a set of digits, and $Y_\Theta$ the associated spectrum. We say that $\Theta$ satisfies the finite type condition (FTC) if $Y_\Theta$ has no accumulation point in $\mathbb{R}$ (c.f. [22, Lemma 2.1]). The purpose of the research is to investigate what characterize the FTC for this class of IFS. In particular, we are interested in whether $q$ has to be a PV number.

We simplify the setting by the following normalization process. Let $\Phi$ be the IFS

$$\Phi \stackrel{\text{def}}{=} \left\{ \rho x + (c_i - c_0)\frac{(1-\rho)}{(c_m - c_0)} : 0 \leq i \leq m \right\} \stackrel{\text{def}}{=} \{\rho x + b_i\}_{i=0}^m .$$

We have

(a) $0 = b_0 < b_1 < \cdots < b_m = 1 - \rho$;

(b) $b_{i+1} - b_i \leq \rho(R - L)\dfrac{(1-\rho)}{(c_m - c_0)} = \rho$;

(c) $[0, 1] = \bigcup_{i=0}^m [\rho \cdot 0 + b_i,\ \rho \cdot 1 + b_i]$.

In addition, if $Y_\Phi$ has no accumulation point in $\mathbb{R}$, then so does $Y_\Theta$, and vice versa. As a result, we can restrict our attention to the class of IFS satisfying condition (a), (b), (c).

It is hoped that through the research, the following objectives can be achieved:

1. To establish some necessary and sufficient conditions for this class of IFS to satisfy the FTC. In particular, to understand what the role of PV numbers is in this problem.

2. To obtain some simple criteria or algorithms for verifying if an IFS of this class satisfies the FTC.

3. To investigate if there are examples of IFS of this class which satisfy the FTC but use non-PV numbers as parameters, and to understand why such examples exist or do not exist.

## 1.3 Main Results and the thesis structure

The starting point of the research is the following findings by Feng (personal communication, 2018):

**Theorem 1.1 (The IFS has algebraic parameters)**
*If $\Phi$ satisfies the FTC, then $q$ is an algebraic integer, $|A| \leq q$ for all algebraic conjugate $A$ of $q$, and $b_i \in \mathbb{Q}(q) = \mathbb{Q}[q]$ for all $i$.*

Here $\mathbb{Q}(q)$ denotes the smallest field containing $q$ and the rational numbers $\mathbb{Q}$, and $\mathbb{Q}[q]$ the ring generated by $q$ over $\mathbb{Q}$. Feng (personal communication, 2018) also suggested that the work [1] by Akiyama and Komornik could be useful. It is indeed so. Based on the aforementioned work, as well as an earlier work [17] by Erdős and Komornik, we are able to say something about our objectives. Recall that $\#\Phi = m + 1$. In this thesis, we demonstrate:

**Theorem 1.2 (Main result)**

I. *If $1 \leq m \leq 3$ and $\Phi$ satisfies the FTC, then $q$ is a PV number.*

II. *If $m = 4$ and $\Phi$ satisfies the FTC, then for any algebraic conjugate $A$ of $q$, $|A| \geq 1$ only if $A$ is not a real number.*

III. *For each $m \geq 4$, there exists $\Phi$ satisfying the FTC, and the associated $q$ is not a PV number.*

To establish this result, we handle the real and complex algebraic conjugates of $q$ separately. We show that:

**Theorem 1.3 (Real cases)**
*Let $A$ be a real algebraic conjugate of $q$. If $\Phi$ satisfies the FTC and $m \leq 4$, then $|A| < 1$.*

**Theorem 1.4 (General cases)**

*Let $A$ be an algebraic conjugate of $q$. Suppose one of the following holds: (i) $b_i/b_m \in \mathbb{Q}$ for all $i$; (ii) $m \leq 3$. If $\Phi$ satisfies the FTC, then $|A| < 1$.*

Observe that items I and II of the main result follow from them. Note also that Theorem 1.4(i) includes the case $\{b_i\}_{i=0}^m = \{(1-\rho)i/m\}_{i=0}^m$ when $(1-\rho)/m \leq \rho$. This situation was considered in [1, 22].

We indeed have more detailed results for the real cases. Let $\partial_i \overset{\text{def}}{=} b_i q$, so that $\Phi = \{\rho x + \partial_i \rho\}_{i=0}^m$. Given an algebraic conjugate $A$ of $q$, let $\sigma_A$: $\mathbb{Q}[q] \to \mathbb{Q}[A]$ be the field isomorphism which maps $q$ to $A$ and leaves $\mathbb{Q}$ fixed. When $A \in \mathbb{R}$, the order relation of $\{\sigma_A(\partial_i)\}_{i=0}^m$ gives rise to a permutation $\mathbb{P} = \mathbb{P}(A)$ of $\{m, m-1, \ldots, 0\}$. Using a code $\langle i_m \cdots i_0 \rangle$ to represent it, we have:

**Theorem 1.5 (Forbidden patterns)**

*Let $A$ be a real algebraic conjugate of $q$. Suppose $\Phi$ satisfies the FTC. If the code representation of $\mathbb{P} = \mathbb{P}(A)$ is in one of the following patterns, then $|A| < 1$.*

(Pa-I) $\langle \cdots 0 \rangle$ *or* $\langle \cdots 1 \rangle$ *or* $\langle \cdots m \rangle$.

(Pa-I′) $\langle m \cdots \rangle$ *or* $\langle (m-1) \cdots \rangle$ *or* $\langle 0 \cdots \rangle$.

(Pa-II) $\langle \ell_0 \cdots m \cdots (\ell_0 + 1) \cdots \rangle$ *for some* $0 \leq \ell_0 \leq m-2$.

(Pa-II′) $\langle \cdots \ell_0 \cdots 0 \cdots (\ell_0 + 1) \rangle$ *for some* $1 \leq \ell_0 \leq m-1$.

$$\text{(Pa-III)} \begin{cases} \langle \cdots (\ell_0+1) \cdots m \cdots \ell_0 \cdots \rangle & \textit{for some } 0 \leq \ell_0 \leq m-1, \textit{ at the same time} \\ \langle \cdots m \cdots j \cdots \rangle & \textit{for all } 0 \leq j \leq \ell_0, \textit{ and} \\ \langle \cdots j \cdots m \cdots \rangle & \textit{for all } \ell_0 + 1 \leq j \leq m. \end{cases}$$

(Pa-IV) $\langle \ell_0 \; 0 \cdots (\ell_0+1) \cdots m \cdots \rangle$ *for some* $0 \leq \ell_0 \leq m-2$.

(Pa-IV′) $\langle \cdots 0 \cdots \ell_0 \cdots m \; (\ell_0+1) \rangle$ *for some* $1 \leq \ell_0 \leq m-1$.

**Theorem 1.6 (“125034”)**

*Let $m = 5$. Let $A$ be a real positive algebraic conjugate of $q$. If $\Phi$ satisfies the FTC, then either $|A| < 1$, or $\mathbb{P} = \mathbb{P}(A)$ has code $\langle 125034 \rangle$.*

**Theorem 1.7 (“120534”)**

*Let $m = 5$. Let $A$ be a real negative algebraic conjugate of $q$. If $\Phi$ satisfies the FTC, then either $|A| < 1$, or $\mathbb{P} = \mathbb{P}(A)$ has code $\langle 120534 \rangle$.*

As Theorem 1.5 suggests, we study pattern-avoiding permutations given by the real numbers $\{\sigma_A(\partial_i)\}_{i=0}^m$ for the real cases.[1] To deal with complex algebraic conjugates, on the other hand, we study pattern-avoiding configurations[2] given by the planar point set $\{\sigma_A(\partial_i)\}_{i=0}^m$. We manage to handle this more complicated problem for $m \leq 3$. We note that $m = 3$ and $m = 4$ are the most intractable cases in this

[1] The term “forbidden pattern” and “pattern-avoiding permutation” come from combinatorics. See e.g. [4] and [56, Chapter 1].

[2] The author coins this term to extend the notion of “pattern-avoiding permutation” to finite point sets in $\mathbb{R}^2$. See e.g. [42] for an introduction to the study of finite point configurations.

research: they are the cases from having a proof to having a counterexample of the necessity of $q$ to be a PV number.

In contrast to items I and II, item III of the main result is demonstrated by using many concrete examples, which are discovered by educated guess and computer experiment. We present:

**Theorem 1.8 (Examples)**
*There exists $\Phi$ satisfying the FTC but the associated $q$ is not a PV number. Using $A$ to denote an algebraic conjugate of $q$, we have the following examples.*

I. *Arbitrary $m \geq 5$, $m$ odd, and $A \in \mathbb{R} \cap (1, q)$.*

II. *Arbitrary $m \geq 5$, $m$ odd, and $A \in \mathbb{R} \cap (-q, -1)$.*

III. *Arbitrary $m \geq 6$, $m$ even, and $A \in \mathbb{R} \cap (1, q)$.*

IV. $m = 4$, $|A| > 1$.

V. $m = 4$, $|A| = 1$.

The above represent our major findings. We should not forget to mention an equivalent formulation of the FTC in terms of the density of $Y$, which was given by Feng [22] through an argument by Drobot [15].

**Theorem 1.9 (Density of $Y$ and FTC)**
*$\Phi$ satisfies the FTC if and only if $Y$ is not dense in $\mathbb{R}$.*

We also remark that in regard to Theorem 1.1, Feng (personal communication, 2021) has proved a stronger result that $q$ is indeed a Perron number[3]. As the proof is involved, we shall not present it for the sake of simplicity.

* * * * * * * * *

We develop the thesis as follows.

- In Chapter 2, "Preparation", we prepare the tools for proving our results. Notation is introduced and the essence of our approach is demonstrated.
- In Chapter 3, "Real cases", we prove our results for the real cases (Theorem 1.3, 1.5-1.7). The establishment of the theorem of "125034" (Theorem 1.6) is a milestone of the research, as it not only reveals the possibility that $q$ need not be a PV number, but also sheds light on how to construct examples. We reveal one such IFS and its properties in an interlude.
- In Chapter 4, "General cases", we prove the theorem of the general cases (Theorem 1.4). In particular, we engage in the intractable case $m = 3$. Also, as a byproduct, we give an isolated result of pattern-avoiding configurations for discrete geometry.

[3]A Perron number is a real algebraic integer which is strictly greater than 1, while all of its algebraic conjugates have absolute values strictly less than it.

- In Chapter 5, "Examples", we demonstrate our examples (Theorem 1.8). As the examples for $m = 4$ play a crucial role in the completeness of our main result, we shall explicate how we discover them. This is the final chapter and we finish by giving miscellaneous results inspired by the examples.

- There are three appendices. In Appendix A, we introduce terminology concerning the FTC and give a proof of Theorem 1.1 and 1.9 respectively. In Appendix B and C, we list some programming codes and output, which are related to the computer experiment of this research.

A word on style. As shown in this section, we shall try to give a caption to every result in this thesis (theorem, lemma, corollary, etc). This is inspired by the writing of Terence Tao.

We finish this chapter by a literature review which can be skipped by the reader.

## 1.4 Literature review

Various separation conditions for IFS having overlaps are proposed in the literature, and the FTC is one of them. These conditions are considered because without imposing certain restriction on the overlapping behavior of the IFS, the analysis of dimensions and multifractal structure is intractable. See [14] for a survey of these conditions. It was shown in [22] that for the class of IFS studied in this research, the FTC is equivalent to the weak separation condition discussed in [38]. In other words, the spectrum $Y_\Phi$ has no accumulation point in $\mathbb{R}$ if and only if 0 is not an accumulation point of $Y_\Phi$.

The work [1] by Akiyama and Komornik implies that for the IFS

$$\Phi := \left\{ \rho x + \frac{i}{m}(1-\rho) :\ 0 \le i \le m \right\},$$

$Y_\Phi$ has no accumulation point in $\mathbb{R}$ if and only if $q := \rho^{-1} \ge m+1$ or $q$ is a PV number. Based on this result, Feng [22] subsequently showed the corresponding result for the density of $Y_\Phi$. These findings suggest that the involvement of PV numbers as parameters seems to play a crucial role in determining whether an IFS can behave in a tidy way. Many research papers reveal this phenomenon (e.g. [1, 10, 17]) or make use of it (e.g. [23, 26, 36, 40]).

In respect of the necessary and sufficient conditions for the class of IFS under consideration to satisfy the FTC, it is known that if $q$ is a PV number and all of the translation parameters $b_i$ are in $\mathbb{Q}[q]$, then the IFS satisfies the FTC [40] (see also [22, Theorem 1.11]). On the other hand, Feng (personal communication, 2021) found that if the FTC holds, then $q$ must be a Perron number and all of the translation parameters $b_i$ are in $\mathbb{Q}[q]$. Although methods of checking whether an algebraic integer is a PV number exist in the literature (e.g. [6, 9]), it seems that in this research they cannot be applied directly.

With respect to the study of fractal geometry, we note that an IFS satisfying the FTC has a remarkable feature: it allows a scheme to compute the Hausdorff dimension of its attractor [40] (see [31, 36, 45] for some special cases). Also, for a homogeneous IFS on $\mathbb{R}$, the FTC implies the multifractal formalism holds for the associated self-similar measures [20, 21].

It is noted that the FTC can be destroyed by a small perturbation of parameters [40]. Therefore, using a computer program to search for IFS satisfying the FTC could be infeasible. Even if an IFS satisfies the FTC so that the number of "neighborhood types" is finite, this number can still be very large (e.g. 4017 in an example of [36]). Without an upper bound of such number, computer could only offer limited assistance in this study.

On the number-theoretic side, researchers of numeration systems investigate issues of unique expansions, finite expansions, periodic expansions, universal expansions, etc, as well as topology properties such as discreteness and denseness of the associated spectra. Ergodic theory also plays a role because when generating an expansion by greedy algorithm for example, we make use of a piecewise linear expanding map on the unit interval. Properties of such map have been well-studied in the literature (e.g. [37]).

Rényi [46] and Parry [43] are pioneers of the number-theoretic and ergodicity aspects of the issue. They considered non-integer bases and integer digits, founding the theory of $\beta$-expansion. Since then, a lot of research has been done on this topic, see for instance [13, 17, 33, 34, 50, 52, 55, 57]. There are also studies dealing with integer bases and non-integer digits (e.g. [11, 31, 41]). Expansions using two bases (corresponding to non-homogeneous IFS with two contractions) are also considered [39]. However, it seems that unlike the fractal geometry side, on the number-theoretic side there is only limited research on both non-integer bases and digits. [12, 35, 36] are some examples.

Besides the base-digit construct, the terms "numeration system" and "number system" indeed have various meanings in the literature, see e.g. [2, 3, 5, 24, 27, 28, 29, 30, 48]. A common feature among them is a systematic way to represent numbers by words of finite or infinite length.

Topics relevant to this research include: polynomials with coefficients from a finite set [8]; eigenvalues of non-negative matrices [32]; Rauzy fractals and central tiles [53]; Salem numbers [54]; morphic words and substitutions [47]. For related topics in algebra, see for instance [16, 19, 49]. For a glimpse of how this problem is related to computer science, see e.g. [7, 25, 27, 44, 47, 48].

Finally, we note the following convention of the symbol used: the base is usually denoted by $\beta$ or $q$ in the literature. The letter $\beta$ is originated from Rényi and is used by papers dealing with ergodic and probabilistic aspects of the issue, while papers dealing with combinatorial and topological aspects use the letter $q$, following Erdős and his collaborators [13].

# Chapter 2

# Preparation

In this chapter, we prepare ourselves for proving the main result. In Section 2.1, we introduce the notation and conventions used throughout the thesis. In Section 2.2, we establish some preliminary results and illustrate the essence of our approach.

## 2.1 Notation and conventions

Let $0 < \rho < 1$. We consider the IFS $\Phi \stackrel{\text{def}}{=} \{\varphi_i(x) = \rho x + b_i\}_{i=0}^m$, where $0 = b_0 < b_1 < \cdots < b_m = 1-\rho$, and $b_{i+1} - b_i \leq \rho$ for all $i$ (we call this the **overlapping hypothesis**). Let $q \stackrel{\text{def}}{=} 1/\rho$. Assume $q$ is an algebraic integer and $b_i \in \mathbb{Q}(q) = \mathbb{Q}[q]$ for all $i$, so that $b_i = \mathsf{g}_i(q)$ for some $\mathsf{g}_i(x) \in \mathbb{Q}[x]$. We also assume that $q \notin \mathbb{N}$, and if $A$ is an algebraic conjugate of $q$, then $|A| \leq q$. We make these assumptions in view of Theorem 1.1 (p.3).

Define $\mathsf{D}(x) \subseteq \mathbb{Q}[x]$ by $\mathsf{D}(x) \stackrel{\text{def}}{=} \{x\mathsf{g}_i(x)\}_{i=0}^m$. Let

$$Y \stackrel{\text{def}}{=} \left\{ \sum_{i=0}^{n} q^i s_i : n \geq 0,\ s_i \in (\mathsf{D}(q) - \mathsf{D}(q)) \right\}.$$

We say that $\Phi$ satisfies the FTC if $Y$ has no accumulation point in $\mathbb{R}$ (c.f. [22, Lemma 2.1]). Let $Y_{\text{core}} \stackrel{\text{def}}{=} Y \cap (0,1]$. We note that $\Phi$ satisfies the FTC if and only if $Y_{\text{core}}$ is a finite set [22].

Given any finite set $\mathbb{E} = \{\delta_1, \ldots, \delta_L\}$ of real numbers such that $\delta_1 < \delta_2 < \cdots < \delta_L$, we define ${\delta_{i+1}}^{(\downarrow\mathbb{E})} = \delta_i$, ${\delta_1}^{(\downarrow\mathbb{E})} = -\infty$. For brevity we shall also write ${\delta_i}^{\downarrow}$ when the underlying set $\mathbb{E}$ is clear. Let $\mathbb{E}^{\times\min} \stackrel{\text{def}}{=} \mathbb{E} \setminus \{\min \mathbb{E}\}$, so that for all $\delta \in \mathbb{E}^{\times\min}$, we have $0 < \delta - \delta^{\downarrow} < \infty$. Let

$$\text{gap}_{\max}(\mathbb{E}) \stackrel{\text{def}}{=} \max\left\{\delta - \delta^{\downarrow} : \delta \in \mathbb{E}^{\times\min}\right\}$$

denote the maximum gap between consecutive elements of $\mathbb{E}$.

We shall use the Vinogradov notation. Given functions $h, k : X \to [0,\infty)$, we write $h \ll k$ if there exists a constant $C > 0$ such that $h(x) \leq Ck(x)$ for all $x \in X$. Write $h \asymp k$ if $h \ll k$ and $k \ll h$.

When using the summation notation $\sum_s^t$, we treat it as an empty sum if the upper limit $t$ is less than the lower limit $s$. For instance, we take $\sum_{i=1}^0 q^{-i} = 0$.

We use $D_R(\xi)$ to denote the closed disc with center $\xi$ and radius $R$ in the complex plane. In particular, $D_1(1) = \{z \in \mathbb{C} : |z-1| \leq 1\}$ is a disc lying entirely in the closed right half plane.

Given an algebraic conjugate $A$ of $q$, define $\sigma_A$: $\mathbb{Q}[q] \to \mathbb{Q}[A]$ to be the field isomorphism which maps $q$ to $A$ and leaves $\mathbb{Q}$ fixed. In addition, given a nonzero complex number $\mathfrak{w} \in \mathbb{C} \setminus \{0\}$, define for each non-negative integer $\ell \geq 0$ the set

$$\mathbb{L}_\ell(A,\mathfrak{w}) \stackrel{\text{def}}{=} \left\{ \delta \in (\mathsf{D}(q) - \mathsf{D}(q)) : \Re\left(\frac{\sigma_A(\delta)\mathfrak{w}}{A^\ell}\right) \geq 0 \right\}.$$

We shall also write $\mathbb{L}_\ell$ for $\mathbb{L}_\ell(A,\mathfrak{w})$ when $A$ and $\mathfrak{w}$ are clear from the context.

We give a code representation to each $\mathbb{L}_\ell(A,\mathfrak{w})$. Write $\Phi \stackrel{\text{def}}{=} \{\rho x + \partial_i \rho\}_{i=0}^m$, so that $\mathsf{D}(q) = \{\partial_i\}_{i=0}^m$. Suppose $i_m \cdots i_0$ is a permutation of $m \cdots 0$ such that

$$\Re\left(\frac{\sigma_A(\partial_{i_m})\mathfrak{w}}{A^\ell}\right) \geq \Re\left(\frac{\sigma_A(\partial_{i_{m-1}})\mathfrak{w}}{A^\ell}\right) \geq \cdots \geq \Re\left(\frac{\sigma_A(\partial_{i_0})\mathfrak{w}}{A^\ell}\right),$$

where if $\Re\left(\sigma_A(\partial_{i_k})\mathfrak{w}/A^\ell\right) = \Re\left(\sigma_A(\partial_{i_{k-1}})\mathfrak{w}/A^\ell\right)$, then we require $i_k > i_{k-1}$. By defining

$$[s;t] \stackrel{\text{def}}{=} \partial_s - \partial_t,$$

we see that $\mathbb{L}_\ell(A,\mathfrak{w}) = \{[i_s;i_t] : m \geq s \geq t \geq 0\}$. In this way, we say that $\mathbb{L}_\ell(A,\mathfrak{w})$ has code $\langle i_m i_{m-1} \cdots i_0 \rangle$. For example, when $m=3$ and $\mathbb{L}_\ell(A,\mathfrak{w})$ has code $\langle 2301 \rangle$, then

$$\Re\left(\frac{\sigma_A(\partial_2)\mathfrak{w}}{A^\ell}\right) > \Re\left(\frac{\sigma_A(\partial_3)\mathfrak{w}}{A^\ell}\right) \geq \Re\left(\frac{\sigma_A(\partial_0)\mathfrak{w}}{A^\ell}\right) > \Re\left(\frac{\sigma_A(\partial_1)\mathfrak{w}}{A^\ell}\right),$$

and $\mathbb{L}_\ell(A,\mathfrak{w}) = \{0, [2;3], [2;0], [2;1], [3;0], [3;1], [0;1]\}$.

Associated with a given $\mathbb{L}_\ell(A,\mathfrak{w})$ which has code representation $\langle i_m \cdots i_0 \rangle$, letting $H_\ell = i_m$ (the "head") we define

$$\begin{aligned} \mathbb{H}_\ell^+ &= \mathbb{H}_\ell^+(A,\mathfrak{w}) \stackrel{\text{def}}{=} \{[H_\ell;i_t] : m \geq t \geq 0\} \stackrel{\text{def}}{=} \{[H_\ell;*]\}, \\ \mathbb{H}_\ell^- &= \mathbb{H}_\ell^-(A,\mathfrak{w}) \stackrel{\text{def}}{=} \{[i_s;H_\ell] : m \geq s \geq 0\} \stackrel{\text{def}}{=} \{[*;H_\ell]\}, \end{aligned}$$

and letting $T_\ell = i_0$ (the "tail"),

$$\begin{aligned} \mathbb{T}_\ell^+ &= \mathbb{T}_\ell^+(A,\mathfrak{w}) \stackrel{\text{def}}{=} \{[i_s;T_\ell] : m \geq s \geq 0\} \stackrel{\text{def}}{=} \{[*;T_\ell]\}, \\ \mathbb{T}_\ell^- &= \mathbb{T}_\ell^-(A,\mathfrak{w}) \stackrel{\text{def}}{=} \{[T_\ell;i_t] : m \geq t \geq 0\} \stackrel{\text{def}}{=} \{[T_\ell;*]\}. \end{aligned}$$

Note that $\mathbb{H}_\ell^+, \mathbb{T}_\ell^+ \subseteq \mathbb{L}_\ell$ and $-\mathbb{H}_\ell^-, -\mathbb{T}_\ell^- \subseteq \mathbb{L}_\ell$. When $A$ is a real algebraic conjugate of $q$ and $\mathfrak{w} = 1$, the set $\mathbb{L}_\ell$ can be described simply. Define

$$\mathbb{P} = \mathbb{P}(A) \stackrel{\text{def}}{=} \mathbb{L}_0(A,1) = \{\delta \in (\mathsf{D}(q) - \mathsf{D}(q)) : \sigma_A(\delta) \geq 0\}.$$

When $A$ is positive, we have $\mathbb{L}_\ell \equiv \mathbb{P}$ for all $\ell \geq 0$, while when $A$ is negative, we have

$$\mathbb{L}_\ell = \begin{cases} \mathbb{P} & \text{if } \ell \text{ is even} \\ -\mathbb{P} & \text{if } \ell \text{ is odd.} \end{cases}$$

We end this section by the following observation:

**Observation 2.1 (Appeal to symmetry)**
*Let $A$ be a real conjugate of $q$. If $\Phi$ satisfies the FTC and $\mathbb{P} = \mathbb{P}(A)$ has code $\langle i_m \cdots i_0 \rangle$, then $\widetilde{\Phi} \stackrel{def}{=} \{\rho x + (1-\rho-b_i)\}_{i=m}^{0}$ also satisfies the FTC, and the associated $\widetilde{\mathbb{P}}(A)$ has code $\langle (m-i_0)\cdots(m-i_m)\rangle$.*

## 2.2 Preliminary results and the essence of the approach

We begin with the following proposition, which is an extension of [17, Lemma 1.4].

**Proposition 2.2 (Consequence of FTC)**
*Let $u_0 \geq 0$ be an integer and $\{s_i(x)\}_{i=-u_0}^{\infty} \subseteq (\mathsf{D}(x) - \mathsf{D}(x))$. Let $A$ be an algebraic conjugate of $q$. Suppose $\Phi$ satisfies the FTC and $\sum_{-u_0}^{\infty} \frac{s_i(q)}{q^i} = 0$. We have*

(a) $\left\{ q^n \sum_{-u_0}^{n} \frac{s_i(q)}{q^i} \right\}_{n \geq 0}$ *is a finite set.*

(b) $\left\{ A^n \sum_{-u_0}^{n} \frac{s_i(A)}{A^i} \right\}_{n \geq 0}$ *is a finite set.*

(c) *If $|A| > 1$, then $\sum_{-u_0}^{\infty} \frac{s_i(A)}{A^i} = 0$.*

**Proof**
By $\sum_{-u_0}^{\infty} \frac{s_i(q)}{q^i} = 0$, we have $\left| q^n \sum_{-u_0}^{n} \frac{s_i(q)}{q^i} \right| = \left| q^n \sum_{n+1}^{\infty} \frac{s_i(q)}{q^i} \right| \ll 1$. Since $Y$ has no accumulation point in $\mathbb{R}$, we get (a). As a result, there exist $n_1, \ldots, n_k \geq 0$ such that

$$\left\{ q^n \sum_{-u_0}^{n} \frac{s_i(q)}{q^i} \right\}_{n \geq 0} = \left\{ q^{n_1} \sum_{-u_0}^{n_1} \frac{s_i(q)}{q^i}, \ldots, q^{n_k} \sum_{-u_0}^{n_k} \frac{s_i(q)}{q^i} \right\}.$$

Hence, given any $N \geq 0$, there exists $n_j$ such that $q^N \sum_{-u_0}^{N} \frac{s_i(q)}{q^i} = q^{n_j} \sum_{-u_0}^{n_j} \frac{s_i(q)}{q^i}$. Note that $x^n \sum_{-u_0}^{n} \frac{s_i(x)}{x^i} \in \mathbb{Q}[x]$ for all $n \geq 0$. Therefore, since $A$ is an algebraic conjugate of $q$, we have $A^N \sum_{-u_0}^{N} \frac{s_i(A)}{A^i} = A^{n_j} \sum_{-u_0}^{n_j} \frac{s_i(A)}{A^i}$, whence

$$\left\{ A^n \sum_{-u_0}^{n} \frac{s_i(A)}{A^i} \right\}_{n \geq 0} = \left\{ A^{n_1} \sum_{-u_0}^{n_1} \frac{s_i(A)}{A^i}, \ldots, A^{n_k} \sum_{-u_0}^{n_k} \frac{s_i(A)}{A^i} \right\},$$

and (b) follows. In particular, $\left| A^n \sum_{-u_0}^{n} \frac{s_i(A)}{A^i} \right| \ll 1$. Hence, if $|A| > 1$, then $\left| \sum_{-u_0}^{n} \frac{s_i(A)}{A^i} \right| \ll |A|^{-n} \to 0$ as $n \to \infty$, which shows (c).

**Q.E.D.**

Next, we present a lazy algorithm which is inspired by [17, Lemma 1.6-1.8].

**Proposition 2.3 (Lazy algorithm)**
*Let $\{\mathbb{E}_i\}_{i=1}^{\infty}$ be a sequence of finite sets of real numbers. Suppose there is a constant $C > 0$ such that $-C \leq \min \mathbb{E}_i \leq \max \mathbb{E}_i \leq C$ for all $i \geq 1$.*

*Let $\sum_{i=1}^{\infty} p_i$ be a convergent series, where $p_i > 0$ for all $i$. Let $x \leq \sum_{i=1}^{\infty} (\max \mathbb{E}_i) p_i$. Define $t_i \in \mathbb{E}_i$ inductively for all $i \geq 1$ as follows. Let $j \geq 1$ and suppose that $t_i$ is already defined for all $1 \leq i < j$. Then define $t_j := \delta$, where $\delta \in \mathbb{E}_j$ satisfies*

$$\sum_{1 \leq i \leq j-1} t_i p_i + \delta^{(\downarrow \mathbb{E}_j)} p_j + \sum_{i \geq j+1} (\max \mathbb{E}_i) p_i < x \leq \sum_{1 \leq i \leq j-1} t_i p_i + \delta p_j + \sum_{i \geq j+1} (\max \mathbb{E}_i) p_i.$$

*We have*

(i) $x \leq \sum_1^{\infty} t_i p_i$.

(ii) *If $t_i \neq \min \mathbb{E}_i$ infinitely often, then $x = \sum_1^{\infty} t_i p_i$.*

(iii) *Suppose for all $\ell \geq 1$, we have*

$$\text{gap}_{\max}(\mathbb{E}_\ell) \leq p_\ell^{-1} \sum_{i=\ell+1}^{\infty} (\max \mathbb{E}_i - \min \mathbb{E}_i) \, p_i. \tag{2.1}$$

*Then for all $x \in [\sum_1^{\infty} (\min \mathbb{E}_i) p_i, \ \sum_{i=1}^{\infty} (\max \mathbb{E}_i) p_i]$, this algorithm gives $x = \sum_1^{\infty} t_i p_i$.*

**Proof**

(i) For all $j \geq 1$ we have $x \leq \sum_1^j t_i p_i + \sum_{j+1}^{\infty} (\max \mathbb{E}_i) p_i$. Letting $j \to \infty$ we get the result.

(ii) Suppose $t_j \neq \min \mathbb{E}_j$. Then $x > \sum_{1 \leq i \leq j-1} t_i p_i + t_j^{\downarrow} p_j + \sum_{i \geq j+1} (\max \mathbb{E}_i) p_i$ with $t_j^{\downarrow} \in \mathbb{E}_j$. Therefore, if $t_j \neq \min \mathbb{E}_j$ infinitely often, then letting $j \to \infty$ through these indices we have $x \geq \sum_1^{\infty} t_i p_i$. Together with (i) we get $x = \sum_1^{\infty} t_i p_i$.

(iii) If on the contrary $x < \sum_1^{\infty} t_i p_i$, then by part (ii) all but a finite number of $t_i$ equal $\min \mathbb{E}_i$. It is impossible that $t_i = \min \mathbb{E}_i$ for all $i \geq 1$, for otherwise $x < \sum_1^{\infty} t_i p_i = \sum_1^{\infty} (\min \mathbb{E}_i) p_i$, contradicting the hypothesis on $x$. Hence there is a last index $j \geq 1$ such that $t_j \neq \min \mathbb{E}_j$. Then

$$\sum_1^{j-1} t_i p_i + t_j p_j + \sum_{j+1}^{\infty} (\min \mathbb{E}_i) p_i = \sum_1^{\infty} t_i p_i > x > \sum_1^{j-1} t_i p_i + t_j^{\downarrow} p_j + \sum_{j+1}^{\infty} (\max \mathbb{E}_i) p_i.$$

However this contradicts (2.1).

**Q.E.D.**

The following result describes the sets $\mathbb{H}_\ell^\pm$ and $\mathbb{T}_\ell^\pm$.

**Lemma 2.4 (Structure of $\mathbb{H}_\ell^\pm$, $\mathbb{T}_\ell^\pm$)**

*For any algebraic conjugate $A$ of $q$, and any nonzero $\mathfrak{w} \in \mathbb{C}\backslash\{0\}$, with $\mathbb{L}_\ell = \mathbb{L}_\ell(A,\mathfrak{w})$, $\mathbb{H}_\ell^\pm = \mathbb{H}_\ell^\pm(A,\mathfrak{w})$, $\mathbb{T}_\ell^\pm = \mathbb{T}_\ell^\pm(A,\mathfrak{w})$, the following hold.*

(a) $\operatorname{gap}_{\max}(\mathbb{H}_\ell^\pm)$, $\operatorname{gap}_{\max}(\mathbb{T}_\ell^\pm) \leq 1$.

(b) $\max \mathbb{H}_\ell^\pm - \min \mathbb{H}_\ell^\pm = q-1 = \max \mathbb{T}_\ell^\pm - \min \mathbb{T}_\ell^\pm$.

**Proof**

Suppose $\mathbb{L}_\ell$ has code $\langle i_m \cdots i_0\rangle$. Write

$$\mathbb{H}_\ell^+ = \{[i_m;0], [i_m;1], \ldots, [i_m;i_m-1], 0, [i_m;i_m+1], [i_m;i_m+2], \ldots, [i_m;m]\}.$$

For $\delta := [i_m;j] \in \mathbb{H}_\ell^+$ with $j < m$, we have $\delta - \delta^{(\downarrow\mathbb{H}_\ell^+)} = [i_m;j] - [i_m;j+1] = [j+1;j] \leq 1$ by the overlapping hypothesis. Also, $\max \mathbb{H}_\ell^+ - \min \mathbb{H}_\ell^+ = [i_m;0] - [i_m;m] = [m;0] = q-1$. The sets $\mathbb{H}_\ell^-, \mathbb{T}_\ell^+, \mathbb{T}_\ell^-$ can be studied similarly.

**Q.E.D.**

Our approach is inspired by [1]. Let us illustrate it by proving a result for a special case.

**Proposition 2.5 (Special case "340512")**

*Let $m = 5$ and $A$ be a real negative algebraic conjugate of $q$. Suppose $\Phi$ satisfies the FTC and $\mathbb{P} = \mathbb{P}(A)$ has code $\langle 340512\rangle$. Then $|A| < 1$.*

**Proof**

It is proved by contradiction. Suppose on the contrary $A \in [-q,-1)$. Let $y := [3;2]$. The code $\langle 340512\rangle$ implies $\sigma_A(y) > 0$ and $\sigma_A(y) \geq \sigma_A([s;t])$ for all $s,t$. Note that $y \neq 1$, for otherwise $1 = \sigma_A([3;2]) \geq \sigma_A([0;5]) = |A-1| > 1$. Hence $y \in (0,1)$ by the overlapping hypothesis. Taking $\mathfrak{w} := 1$, $\mathbb{L}_i = \mathbb{L}_i(A,\mathfrak{w})$ and $\mathbb{H}_i^- = \mathbb{H}_i^-(A,\mathfrak{w})$, we let $k_0 = k_0(y)$ be the smallest non-negative integer such that

$$y \leq \sum_{i=1}^{k_0} \frac{q-1}{q^i} + \sum_{i=k_0+1}^{\infty} \frac{\max \mathbb{H}_i^-}{q^i}.$$

Let $k_0^* := \begin{cases} 1 & \text{if } k_0 = 0 \\ k_0 & \text{otherwise} \end{cases}$. We now define for each $i \geq k_0^*$ a set $\mathbb{E}_i \subseteq (\mathsf{D}(q) - \mathsf{D}(q))$. If $k_0 = 0$, then we define $\mathbb{E}_i := \mathbb{H}_i^-$ for all $i \geq k_0^*$. Else if $k_0 \geq 1$, then we define $\mathbb{E}_i$ by

$$\mathbb{E}_i := \begin{cases} \{[5;0], [5;1], [5;2], [4;2], [3;2], 0\} =: \mathbb{S}_- & \text{if } i = k_0^* \text{ and } k_0^* \text{ is odd} \\ \{[5;0], [4;0], [3;0], [2;0], [1;0], 0\} =: \mathbb{S}_+ & \text{if } i = k_0^* \text{ and } k_0^* \text{ is even} \\ \mathbb{H}_i^- & \text{if } i > k_0^*. \end{cases}$$

Noting that $q-1 \in \mathbb{S}_\pm$, we have

$$y - \sum_{i=1}^{k_0^*-1} \frac{q-1}{q^i} \le \sum_{i=k_0^*}^{\infty} \frac{\max \mathbb{E}_i}{q^i}.$$

As $0 \in \mathbb{H}_i^-$ and $\min \mathbb{S}_\pm \le 0$, by $y > 0$ and the definition of $k_0$ we also have

$$\begin{aligned} y - \sum_{i=1}^{k_0^*-1} \frac{q-1}{q^i} &> \begin{cases} 0 & \text{if } k_0 = 0 \\ \left( \sum_{i=1}^{k_0^*-1} \frac{q-1}{q^i} + \sum_{i=k_0^*}^{\infty} \frac{\max \mathbb{H}_i^-}{q^i} \right) - \sum_{i=1}^{k_0^*-1} \frac{q-1}{q^i} & \text{if } k_0 \ge 1 \end{cases} \\ &\ge \sum_{i=k_0^*}^{\infty} \frac{\min \mathbb{E}_i}{q^i}. \end{aligned}$$

Moreover, we have

$$\mathrm{gap}_{\max}(\mathbb{E}_\ell) \le 1 = q^\ell \sum_{i=\ell+1}^{\infty} \frac{\max \mathbb{E}_i - \min \mathbb{E}_i}{q^i}$$

for all $\ell \ge k_0^*$, by using the definition of $\mathbb{S}_\pm$ and the structure of $\mathbb{H}_i^-$ (Lemma 2.4). Thus, our lazy algorithm (Proposition 2.3), applied to $x := y - \sum_1^{k_0-1} \frac{q-1}{q^i}$, gives

$$y - \sum_1^{k_0-1} \frac{q-1}{q^i} = \sum_{i=k_0^*}^{\infty} \frac{t_i}{q^i}$$

with $t_i \in \mathbb{E}_i$ for all $i \ge k_0^*$. Noting that $y \in Y$ and $|A| > 1$, we have

$$\sigma_A(y) = \sum_1^{k_0-1} \frac{A-1}{A^i} + \sum_{i=k_0^*}^{\infty} \frac{\sigma_A(t_i)}{A^i} \tag{2.2}$$

as a consequence of the FTC (Proposition 2.2).

We now show that this is a contradiction. Recall that $\sigma_A(y) \ge \sigma_A([0;5]) > 0$ and $-\mathbb{H}_i^- \subseteq \mathbb{L}_i$. If $k_0 = 0$, then by (2.2), $0 < \sigma_A(y) = \sum_{i=1}^{\infty} \sigma_A(t_i)/A^i \le 0$, giving a contradiction. Else if $k_0 \ge 1$, (2.2) gives

$$\sigma_A(y) \le \sum_{i=1}^{k_0-1} \frac{(A-1)}{A^i} + \frac{\sigma_A(t_{k_0})}{A^{k_0}} = 1 - \frac{1}{A^{k_0-1}} + \frac{\sigma_A(t_{k_0})}{A^{k_0}}. \tag{2.3}$$

Suppose $k_0$ is odd first, so that $\mathbb{E}_{k_0} = \mathbb{S}_-$. By the definition of $\mathbb{S}_-$ and the code of $\mathbb{P}$, for any $\delta \in \mathbb{E}_{k_0}$, $\sigma_A(\delta) < 0$ only if $\delta = [5;0]$. Consequently, it follows from $\sigma_A(y) \ge \sigma_A([0;5])$ and (2.3) that

$$1 - A = \sigma_A([0;5]) \le \sigma_A(y) \le 1 - \frac{1}{A^{k_0-1}} + \frac{\sigma_A([5;0])}{A^{k_0}} = 1 - \frac{1}{A^{k_0}} \le 2,$$

whence $-1 \le A$, which is a contradiction. Therefore, $k_0$ is not odd but even, so that $\mathbb{E}_{k_0} = \mathbb{S}_+$. By the code of $\mathbb{P}$, we see that $\max \mathbb{E}_{k_0} = [3;0]$. Since $\sigma_A(y) \ge \sigma_A([3;5])$, (2.3) gives

$$\sigma_A(\partial_3) - (A-1) = \sigma_A([3;5]) \le \sigma_A(y) \le 1 - \frac{1}{A^{k_0-1}} + \frac{\sigma_A(\partial_3)}{A^{k_0}}.$$

Noting that $\sigma_A(\partial_3) \geq 0$ by the code representation of $\mathbb{P}$, the above gives

$$0 \leq \sigma_A(\partial_3)\left(1 - \frac{1}{A^{k_0}}\right) \leq A - \frac{1}{A^{k_0-1}} < -1 + 1 = 0.$$

This is a contradiction and completes the proof.

**Q.E.D.**

In the same spirit, we prove the following result.

**Proposition 2.6 (Sufficient conditions for $|A| \leq 1$)**
*Let $A$ be an algebraic conjugate of $q$. Suppose $\Phi$ satisfies the FTC. Also, suppose the following hold:*

1. *There exists $y \in Y \cap (0,1)$ and $\mathfrak{w} \in \mathbb{C} \setminus \{0\}$ such that $\Re\left(\sigma_A(y)\mathfrak{w}\right) < 0$.*
2. *For each $\ell \geq 1$, we have $\Re\left(\sum_{i=1}^{\ell-1} \frac{(A-1)\mathfrak{w}}{A^i}\right) \geq 0$.*
3. *For each $\ell \geq 1$, there exists $\mathbb{S}_\ell \subseteq (\mathsf{D}(q) - \mathsf{D}(q))$ such that*
   (a) *$q-1 \in \mathbb{S}_\ell$ and $\min \mathbb{S}_\ell \leq 0$;*
   (b) *$\mathrm{gap}_{\max}(\mathbb{S}_\ell) \leq 1$;*
   (c) *for all $\delta \in \mathbb{S}_\ell$, at least one of the following holds:*
      i. $\Re\left(\frac{\sigma_A(\delta)\mathfrak{w}}{A^\ell}\right) \geq 0.$
      ii. $\Re\left(\frac{\sigma_A(\delta)\mathfrak{w}}{A^\ell}\right) \geq \Re\left(\frac{(A-1)\mathfrak{w}}{A^\ell}\right).$
      iii. $\left|\Re\left(\sigma_A(y)\mathfrak{w}\right)\right| > \left|\Re\left(\frac{\sigma_A(\delta)\mathfrak{w}}{A^\ell}\right)\right|.$
      iv. $\Re\left(\sum_{i=1}^{\ell-1} \frac{(A-1)\mathfrak{w}}{A^i} + \frac{\sigma_A(\delta)\mathfrak{w}}{A^\ell}\right) > \Re\left(\sigma_A(y)\mathfrak{w}\right).$

*Then $|A| \leq 1$.*

**Proof**
It is proved by contradiction. Suppose on the contrary $|A| > 1$. Take $\mathbb{L}_i = \mathbb{L}_i(A, \mathfrak{w})$ and $\mathbb{H}_i^+ = \mathbb{H}_i^+(A, \mathfrak{w})$. As $y \in (0,1)$, let $k_0$ be the smallest non-negative integer such that

$$y \leq \sum_{i=1}^{k_0} \frac{q-1}{q^i} + \sum_{i=k_0+1}^{\infty} \frac{\max \mathbb{H}_i^+}{q^i}.$$

Let $k_0^* := \begin{cases} 1 & \text{if } k_0 = 0 \\ k_0 & \text{otherwise} \end{cases}$. For each $i \geq k_0^*$, we define a set $\mathbb{E}_i \subseteq (\mathsf{D}(q) - \mathsf{D}(q))$. If $k_0 = 0$, then we define $\mathbb{E}_i := \mathbb{H}_i^+$ for all $i \geq k_0^*$. Else if $k_0 \geq 1$, we take

$$\mathbb{E}_i := \begin{cases} \mathbb{S}_i & \text{if } i = k_0^* \\ \mathbb{H}_i^+ & \text{if } i > k_0^*. \end{cases}$$

Since $q-1 \in \mathbb{S}_{k_0^*}$, $0 \in \mathbb{H}_i^+$ and $\min \mathbb{S}_{k_0^*} \leq 0$, by the definition of $k_0$ we have

$$\sum_{i=k_0^*}^{\infty} \frac{\min \mathbb{E}_i}{q^i} < y - \sum_{i=1}^{k_0^*-1} \frac{q-1}{q^i} \leq \sum_{i=k_0^*}^{\infty} \frac{\max \mathbb{E}_i}{q^i}.$$

Also, by hypothesis 3(b) and the structure of $\mathbb{H}_i^+$ (Lemma 2.4), condition (2.1) of our lazy algorithm (Proposition 2.3) is satisfied. Therefore, applying that result to $x := y - \sum_1^{k_0-1} \frac{q-1}{q^i}$, we get

$$y - \sum_{1}^{k_0-1} \frac{q-1}{q^i} = \sum_{i=k_0^*}^{\infty} \frac{t_i}{q^i}$$

with $t_i \in \mathbb{E}_i$ for all $i \geq k_0^*$. As $y \in Y$, it follows from the FTC and Proposition 2.2 that

$$\sigma_A(y) = \sum_{1}^{k_0-1} \frac{A-1}{A^i} + \sum_{i=k_0^*}^{\infty} \frac{\sigma_A(t_i)}{A^i}. \tag{2.4}$$

We now show that this is a contradiction. By hypothesis 1, we have $\Re\left(\sigma_A(y)\mathfrak{w}\right) < 0$. Since $\mathbb{H}_i^+ \subseteq \mathbb{L}_i$, so if $k_0 = 0$, then by (2.4), we have $0 > \Re\left(\sigma_A(y)\mathfrak{w}\right) = \sum_{i=1}^{\infty} \Re\left(\sigma_A(t_i)\mathfrak{w}/A^i\right) \geq 0$, giving a contradiction. Else if $k_0 \geq 1$, (2.4) gives

$$0 > \Re\left(\sigma_A(y)\mathfrak{w}\right) \geq \Re\left(\sum_{i=1}^{k_0-1} \frac{(A-1)\mathfrak{w}}{A^i}\right) + \Re\left(\frac{\sigma_A(t_{k_0})\mathfrak{w}}{A^{k_0}}\right).$$

Since $t_{k_0} \in \mathbb{S}_{k_0}$, the last expression is

$$\geq \begin{cases} \Re\left(\sum_{i=1}^{k_0-1} \frac{(A-1)\mathfrak{w}}{A^i}\right) & \text{if } t_{k_0} \text{ satisfies hypothesis 3(c)(i)} \\ \Re\left(\sum_{i=1}^{k_0} \frac{(A-1)\mathfrak{w}}{A^i}\right) & \text{if } t_{k_0} \text{ satisfies hypothesis 3(c)(ii)} \\ \Re\left(\sum_{i=1}^{k_0-1} \frac{(A-1)\mathfrak{w}}{A^i}\right) + \Re\left(\frac{\sigma_A(t_{k_0})\mathfrak{w}}{A^{k_0}}\right) & \text{if } t_{k_0} \text{ satisfies hypothesis 3(c)(iii) or (iv).} \end{cases}$$

The first and the second are $\geq 0$ by hypothesis 2, which contradict $\Re\left(\sigma_A(y)\mathfrak{w}\right) < 0$. The third plainly contradicts hypothesis 3(c)(iv). Also, by hypothesis 2, it implies

$$0 > \Re\left(\sigma_A(y)\mathfrak{w}\right) \geq -\left|\Re\left(\frac{\sigma_A(t_{k_0})\mathfrak{w}}{A^{k_0}}\right)\right|,$$

which contradicts hypothesis 3(c)(iii).

**Q.E.D.**

The rest of the section is devoted to finding suitable $y \in Y \cap (0,1)$ satisfying condition 1 of the preceding proposition. The idea was suggested by Feng (personal communication, 2020).

**Lemma 2.7 (Uniform upper bound of normalized gaps)**

*Fix* $\theta_0 \in (0,1]$. *Let* $F_n := \left\{\sum_1^n \frac{d_i}{q^i} : d_i \in \mathsf{D}(q)\right\}$. *If* $\mathrm{gap}_{\max}(\mathsf{D}(q)) \leq \theta_0$, *then* $\mathrm{gap}_{\max}(q^n F_n) \leq \theta_0$ *for all* $n \geq 1$. *In particular, by the overlapping hypothesis, we have* $\mathrm{gap}_{\max}(q^n F_n) \leq 1$ *for all* $n \geq 1$.

**Proof**

It is proved by induction on $n$. When $n = 1$, the statement is true since $qF_1 = \mathsf{D}(q)$. Suppose the statement is true for all $n \leq k$. Let $f \in F_{k+1}^{\times \min}$. We want to show that $q^{k+1}(f - f^{(\downarrow F_{k+1})}) \leq \theta_0$.

Writing $f := \sum_{i=1}^{k+1} d_i/q^i$, there exists an index $j_0$, $1 \le j_0 \le k+1$, such that $d_{j_0} \neq 0$ and $d_i = 0$ for all $j_0 < i \le k+1$. With $d_{j_0}^{\downarrow} = d_{j_0}^{(\downarrow \mathsf{D}(q))}$, we have

$$\begin{aligned} f - \left[ \sum_{i=1}^{j_0-1} \frac{d_i}{q^i} + \frac{d_{j_0}^{\downarrow}}{q^{j_0}} + \sum_{j_0+1}^{k+1} \frac{q-1}{q^i} \right] &= \frac{d_{j_0} - d_{j_0}^{\downarrow}}{q^{j_0}} - \left( \frac{1}{q^{j_0}} - \frac{1}{q^{k+1}} \right) \\ &\le \frac{\theta_0}{q^{j_0}} - \left( \frac{1}{q^{j_0}} - \frac{1}{q^{k+1}} \right) \le \frac{\theta_0}{q^{k+1}}. \end{aligned}$$

Thus, if the L.H.S. is $> 0$, then

$$q^{k+1}(f - f^{\downarrow}) \le q^{k+1} \left( f - \left[ \sum_{1}^{j_0-1} \frac{d_i}{q^i} + \frac{d_{j_0}^{\downarrow}}{q^{j_0}} + \sum_{j_0+1}^{k+1} \frac{q-1}{q^i} \right] \right) \le \theta_0,$$

which was to be shown. Else, we have

$$\sum_{i=1}^{j_0-1} \frac{d_i}{q^i} + \frac{d_{j_0}^{\downarrow}}{q^{j_0}} < f \le \sum_{i=1}^{j_0-1} \frac{d_i}{q^i} + \frac{d_{j_0}^{\downarrow}}{q^{j_0}} + \sum_{j_0+1}^{k+1} \frac{q-1}{q^i}$$

and $j_0 \neq k+1$. This implies

$$\min F_{k+1-j_0} = 0 < q^{j_0} \left( f - \sum_{i=1}^{j_0-1} \frac{d_i}{q^i} - \frac{d_{j_0}^{\downarrow}}{q^{j_0}} \right) \le q^{j_0} \sum_{j_0+1}^{k+1} \frac{q-1}{q^i} = \max F_{k+1-j_0},$$

so there exists $g \in F_{k+1-j_0}^{\times \min}$ such that, with $g^{\downarrow} = g^{(\downarrow F_{k+1-j_0})}$,

$$g^{\downarrow} < q^{j_0} \left( f - \sum_{i=1}^{j_0-1} \frac{d_i}{q^i} - \frac{d_{j_0}^{\downarrow}}{q^{j_0}} \right) \le g.$$

As $\left[ \sum_{i=1}^{j_0-1} \frac{d_i}{q^i} + \frac{d_{j_0}^{\downarrow}}{q^{j_0}} + \frac{g^{\downarrow}}{q^{j_0}} \right] \in F_{k+1}$, it follows that

$$q^{k+1}(f - f^{\downarrow}) \le q^{k+1} \left( f - \left[ \sum_{i=1}^{j_0-1} \frac{d_i}{q^i} + \frac{d_{j_0}^{\downarrow}}{q^{j_0}} + \frac{g^{\downarrow}}{q^{j_0}} \right] \right) \le q^{k+1} \left( \frac{g}{q^{j_0}} - \frac{g^{\downarrow}}{q^{j_0}} \right),$$

which is $\le \theta_0$ by the induction hypothesis. This completes the proof.

**Q.E.D.**

**Proposition 2.8 (Availability of $y$)**

*Let $A$ be an algebraic conjugate of $q$. Suppose $\Phi$ satisfies the FTC.*

(a) *Given any nonzero $\xi \in \mathbb{C} \setminus \{0\}$, there exists $y \in Y_{\text{core}}$ such that $\Re(\sigma_A(y)\xi) \le 0$.*

(b) *If $A$ is real, then there exists $y \in Y \cap (0,1)$ such that $\sigma_A(y) < 0$.*

(c) *If $\Re(A) < 1$, then there exists $y \in Y \cap (0,1)$ such that $\Re(\sigma_A(y)) < 0$.*

(d) *There exists $y \in Y \cap (0,1)$ such that given any small $\varepsilon > 0$, there exists $\varpi_\varepsilon \in \mathbb{C} \setminus \{0\}$ with $|\varpi_\varepsilon| = 1$ satisfying the following. (i) $|\varpi_\varepsilon - 1| < \varepsilon$; (ii) $\Re(\sigma_A(y)\varpi_\varepsilon) < 0$; and (iii) $\Re(\sigma_A(y)\varpi_\varepsilon) < \Re(\zeta\varpi_\varepsilon)$ for all $\zeta$ in the closed disc $D_1(1)$.*

**Proof**

(a) It is proved by contradiction. Suppose on the contrary $\Re\left(\sigma_A(y)\xi\right) > 0$ for all $y \in Y_{\text{core}}$. For the $F_n$ defined in the preceding Lemma 2.7, we have $q^n\left[f - f^{(\downarrow F_n)}\right] \in Y_{\text{core}}$ for all $n \geq 1$ and all $f \in F_n^{\times\min}$. Since $\Phi$ satisfies the FTC, $Y_{\text{core}}$ and hence $\xi \cdot \sigma_A\left(Y_{\text{core}}\right)$ are both finite sets. Therefore, we have $q^n\left[f - f^{\downarrow}\right] \asymp 1$ and $\Re\left(\xi \cdot \sigma_A\left(q^n\left[f - f^{\downarrow}\right]\right)\right) \asymp 1$ by our initial assumption.

Now

$$1 - \frac{1}{q^n} = \sum_1^n \frac{q-1}{q^i} = \sum_{f \in F_n^{\times\min}} \left[f - f^{\downarrow}\right] \asymp q^{-n}\left(\left(\#F_n\right) - 1\right).$$

Consequently,

$$\begin{aligned}\Re\left(\left(A^n - 1\right)\xi\right) &= \Re\left(\xi A^n \cdot \sigma_A\left(\sum_{f \in F_n^{\times\min}} \left[f - f^{\downarrow}\right]\right)\right) \\ &= \sum_{f \in F_n^{\times\min}} \Re\left(\xi \cdot \sigma_A\left(q^n\left[f - f^{\downarrow}\right]\right)\right) \gg \sum_{f \in F_n^{\times\min}} 1 \gg q^n,\end{aligned}$$

whence

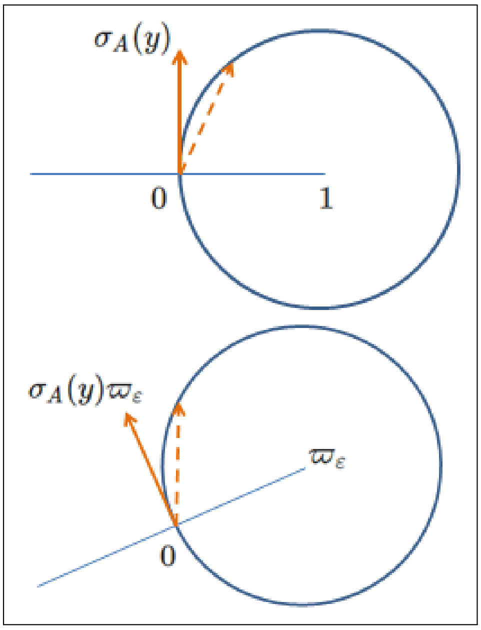
$$\Re\left(\frac{A^n - 1}{A^n}\frac{A^n}{q^n}\xi\right) \gg 1.$$

If $|A| < q$, then by letting $n \to \infty$ we see a contradiction. Else if $|A| = q$, then $\Re\left(\frac{A^n}{q^n}\xi\right) \leq 0$ for infinitely many $n$. Since $\frac{A^n - 1}{A^n} \to 1$ as $n \to \infty$, and $\left|\frac{A^n}{q^n}\xi\right| \equiv |\xi|$, so given any $\varepsilon > 0$, we have $\Re\left(\frac{A^n - 1}{A^n}\frac{A^n}{q^n}\xi\right) < \varepsilon$ for infinitely many $n$. This demonstrates a contradiction.

(b) Taking $\xi = 1$ in part (a), there exists $y \in Y \cap (0,1]$ such that $\sigma_A(y) = \Re\left(\sigma_A(y)\xi\right) \leq 0$. Since $\sigma_A(1) = 1$ and $\sigma_A(x) = 0 \Leftrightarrow x = 0$, we have $y \neq 1$ and $\sigma_A(y) < 0$.

(c) As $0 > \Re\left(A - 1\right) = \Re\left(\sigma_A(\sum_{i=1}^m [i; i-1])\right) = \sum_{i=1}^m \Re\left(\sigma_A([i; i-1])\right)$, there exists $i_0$ such that $\Re\left(\sigma_A([i_0; i_0 - 1])\right) < 0$. Therefore we can take $y := [i_0; i_0 - 1]$.

(d) Taking $\xi = 1$ in part (a), there exists $y \in Y \cap (0,1)$ such that $\Re\left(\sigma_A(y)\right) \leq 0$. If $\Re\left(\sigma_A(y)\right) < 0$, then we can simply take $\varpi_\varepsilon := 1$ and the result follows. Suppose $\Re\left(\sigma_A(y)\right) = 0$. Since $y \neq 0$, so is $\sigma_A(y)$. Without loss of generality we assume $\Im\left(\sigma_A(y)\right) > 0$ so that $\sigma_A(y) = i\left|\sigma_A(y)\right|$. The following figure may explain the subsequent argument clearer than words:




We define a number $\theta_0$ as follows. If $|\sigma_A(y)| < 2$, then define $\theta_0$ to be the number in $(0, \pi/2]$ such that $\sigma_A(y)e^{-i\theta_0}$ is on the circle $|z-1| = 1$. Else if $|\sigma_A(y)| \geq 2$, then define $\theta_0$ to be $\pi/2$. Consider $\varpi_\varepsilon := e^{i\theta}$, where $\theta \in (0, \theta_0)$ is a small number such that $\left|e^{i\theta} - 1\right| < \varepsilon$. We have $\Re\left(\sigma_A(y)\varpi_\varepsilon\right) = -\left|\sigma_A(y)\right| \sin\theta < 0$.

Let $\zeta := re^{i\alpha} \in D_1(1)$, where $-\pi/2 \leq \alpha \leq \pi/2$. Note that if $\alpha = \pm\pi/2$, then $\zeta = 0$, forcing $r = 0$. We want to show that $\Re\left(\zeta\varpi_\varepsilon\right) > \Re\left(\sigma_A(y)\varpi_\varepsilon\right)$. Observe that in the preceding figure, the dashed arrow splits $D_1(1)$ into two regions. They correspond to two cases:

(i) $\alpha + \theta \in [-\pi/2, \pi/2]$.

(ii) $\alpha + \theta > \pi/2$.

In the former case, we have $\Re\left(\zeta\varpi_\varepsilon\right) = r\cos(\alpha+\theta) \geq 0 > \Re\left(\sigma_A(y)\varpi_\varepsilon\right)$. In the latter case, as the figure depicts, we have $r \leq |\sigma_A(y)|$. Therefore,

$$\begin{aligned}\Re\left(\zeta\varpi_\varepsilon\right) = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta &\geq \begin{cases} 0 & \text{if } \alpha = \pm\pi/2 \\ -\left|r\sin\alpha\right|\sin\theta & \text{otherwise} \end{cases} \\ &> -\left|\sigma_A(y)\right|\sin\theta = \Re\left(\sigma_A(y)\varpi_\varepsilon\right).\end{aligned}$$

This completes the proof.

**Q.E.D.**

This is the end of our preparation.

# Chapter 3

# Real cases

In this chapter, we prove our results for the real cases. As revealed by the special case $\langle 340512\rangle$ (p.13 Proposition 2.5), forbidden patterns and their avoidance will be the central theme. Another focus is to illustrate what we can learn from the results. It turns out that we are able to give an example of $\Phi$ satisfying the FTC while the associated $q$ is not a PV number.

## 3.1 Proof of the real cases

We first establish the theorem of forbidden patterns, which plays a principal role for the real cases.

**Proof of the forbidden patterns (p.4 Theorem 1.5)**

It is proved by contradiction using p.15 Proposition 2.6. Suppose on the contrary $A \in [-q,-1) \cup (1,q)$. We take $\mathfrak{w} := 1$. Since $\sum_{i=1}^{\ell-1} \frac{(A-1)}{A^i} = 1 - \frac{1}{A^{\ell-1}} \geq 0$, Proposition 2.6(2) is satisfied. To get a contradiction, we would like to find a positive number $y$ and two sets $\mathbb{S}_+, \mathbb{S}_- \subseteq (\mathsf{D}(q){-}\mathsf{D}(q))$ satisfying the following conditions:

C1. $y \in Y \cap (0,1)$ and $\sigma_A(y) < 0$.

C2. (a) $q-1 \in \mathbb{S}_\pm$ and $\min \mathbb{S}_\pm \leq 0$.

(b) $\operatorname{gap}_{\max}(\mathbb{S}_\pm) \leq 1$.

C3. For all $\delta \in \mathbb{S}_+$, at least one of the following holds:

(a) $\sigma_A(\delta) \geq 0$.

(b) $\sigma_A(\delta) \geq \sigma_A(q-1)$.

(c) $|\sigma_A(y)| > \left|\frac{\sigma_A(\delta)}{A}\right|$.

(d) $1 - \frac{1}{A^{\ell-1}} + \frac{\sigma_A(\delta)}{A^\ell} > \sigma_A(y)$ for all $\ell \geq 1$.

C4. For all $\delta \in \mathbb{S}_-$, at least one of the following holds:

(a) $\sigma_A(\delta) \leq 0$.

(b) $\sigma_A(\delta) \leq \sigma_A(q-1)$.

(c) $|\sigma_A(y)| > \left|\frac{\sigma_A(\delta)}{A}\right|$.

(d) $1 - \frac{1}{A^{\ell-1}} + \frac{\sigma_A(\delta)}{A^\ell} > \sigma_A(y)$ for all $\ell \geq 1$.

And we can assume $A < 0$ when checking C4.

Before proceeding, note that by p.17 Proposition 2.8(b), there exists $\gamma_0 \in Y \cap (0,1)$ such that $\sigma_A(\gamma_0) < 0$. Also, by an appeal to symmetry (p.11 Observation 2.1), if the result for Pa-I (resp. Pa-II, Pa-IV) is established, then so is that for Pa-I′ (resp. Pa-II′, Pa-IV′).

*Pa-I:*

- If $\mathbb{P}$ has code $\langle \cdots 0\rangle$, then we take $y := \gamma_0$, $\mathbb{S}_+ := \{[*;0]\}$ and $\mathbb{S}_- := \{[m;*]\}$, i.e.

$$\mathbb{S}_+ = \{[m;0], [m-1;0], \ldots, [0;0]\},$$
$$\mathbb{S}_- = \{[m;0], [m;1], \ldots, [m;m]\}.$$

  As $q-1 = [m;0] \in \mathbb{S}_\pm$ and $0 = [0;0] = [m;m] \in \mathbb{S}_\pm$, C2(a) is satisfied. C2(b) follows from the overlapping hypothesis. Since $\mathbb{P}$ has code $\langle \cdots 0\rangle$, C3(a) is satisfied. Finally, given $\delta := [m;j] \in \mathbb{S}_-$,

$$\begin{aligned} \mathbb{P} \text{ has code } \langle \cdots j \cdots m \cdots 0\rangle &\;\Rightarrow\; \sigma_A(\delta) \le 0, \\ \text{while } \mathbb{P} \text{ has code } \langle \cdots m \cdots j \cdots 0\rangle &\;\Rightarrow\; \sigma_A([m;0]) \ge \sigma_A([m;j]) = \sigma_A(\delta). \end{aligned}$$

  Consequently, C4(a) or (b) is satisfied.

- If $\mathbb{P}$ has code $\langle \cdots 1\rangle$, then we take $y := [1;0]$, $\mathbb{S}_+ := \{q-1\} \cup \{[*;1]\} = \{[m;0], [*;1]\}$, and $\mathbb{S}_- := \{[m;*]\}$. Since $\mathbb{P}$ has code $\langle \cdots 0 \cdots 1\rangle$, we have $\sigma_A(\partial_0) > \sigma_A(\partial_1)$, whence $\sigma_A(y) < 0$ and so $y \ne 1$. Therefore, by the overlapping hypothesis, $y \in Y \cap (0,1)$, and C1 is satisfied. So is C2, because for $\mathbb{S}_+$ we have $0 < [m;0] - [m;1] = [1;0] \le 1$.

  Note that given $\delta \in \mathbb{S}_+$, either $\delta = q-1$ so that C3(b) is satisfied, or $\delta \in \{[*;1]\} \subseteq \mathbb{P}$ so that C3(a) is satisfied. Finally, to check C4, recall that we can assume $A < 0$. As a result, $\mathbb{P}$ has code $\langle \cdots 0 \cdots m \cdots 1\rangle$, whence given $\delta := [m;j] \in \mathbb{S}_-$,

$$\begin{aligned} \mathbb{P} \text{ has code } \langle \cdots j \cdots m \cdots 1\rangle &\;\Rightarrow\; \sigma_A(\delta) \le 0, \\ \text{while } \mathbb{P} \text{ has code } \langle \cdots 0 \cdots m \cdots j \cdots 1\rangle &\;\Rightarrow\; |\sigma_A(\delta)| \le |\sigma_A(y)| < |A| \cdot |\sigma_A(y)|. \end{aligned}$$

  Thus C4(a) or (c) is satisfied.

- If $\mathbb{P}$ has code $\langle \cdots m\rangle$, then we take $y := \gamma_0$, $\mathbb{S}_+ := \{[*;0]\}$, and $\mathbb{S}_- := \{[m;*]\}$. Plainly C2 is satisfied. Given $[j;0] \in \mathbb{S}_+$,

$$\begin{aligned} \mathbb{P} \text{ has code } \langle \cdots j \cdots 0 \cdots m\rangle &\;\Rightarrow\; \sigma_A([j;0]) \ge 0, \\ \text{while } \mathbb{P} \text{ has code } \langle \cdots 0 \cdots j \cdots m\rangle &\;\Rightarrow\; 0 \ge \sigma_A([j;0]) \ge \sigma_A([m;0]). \end{aligned}$$

  Hence C3(a) or (b) is satisfied. Since $\mathbb{S}_- \subseteq -\mathbb{P}$, C4(a) is satisfied.

This completes the proof for Pa-I.

*Pa-II:* Suppose $\mathbb{P}$ has code $\langle \ell_0 \cdots m \cdots (\ell_0+1) \cdots\rangle$ for some $0 \le \ell_0 \le m-2$. Take $y := [\ell_0+1; \ell_0]$, which fulfills C1. We use $\mathbb{S}_+ := \{[m;*]\}$ and $\mathbb{S}_- = \{[*;0]\}$. Given $\delta := [m;j] \in \mathbb{S}_+$,

$$\begin{aligned} \mathbb{P} \text{ has code } \langle \cdots m \cdots j \cdots\rangle &\;\Rightarrow\; \sigma_A(\delta) \ge 0, \\ \text{while } \mathbb{P} \text{ has code } \langle \ell_0 \cdots j \cdots m \cdots (\ell_0+1) \cdots\rangle &\;\Rightarrow\; |\sigma_A(\delta)| \le |\sigma_A(y)|. \end{aligned}$$

Hence C3(a) or (c) is satisfied. Finally, to check C4, recall that we can assume $A < 0$, whence $\mathbb{P}$ has code $\langle \ell_0 \cdots 0 \cdots m \cdots (\ell_0 + 1) \cdots \rangle$. Given $\delta := [j; 0] \in \mathbb{S}_-$,

$$\begin{aligned} \mathbb{P} \text{ has code } \langle \cdots 0 \cdots j \cdots \rangle \quad &\Rightarrow \quad \sigma_A(\delta) \leq 0, \\ \text{while } \mathbb{P} \text{ has code } \langle \ell_0 \cdots j \cdots 0 \cdots m \cdots (\ell_0 + 1) \cdots \rangle \quad &\Rightarrow \quad |\sigma_A(\delta)| \leq |\sigma_A(y)|. \end{aligned}$$

We see that C4(a) or (c) is satisfied.

*Pa-III:* Take $y := \gamma_0$,

$$\mathbb{S}_+ := \{[m; 0], [m; 1], \ldots, [m; \ell_0]\} \ \cup \ \{[m-1; \ell_0], [m-2; \ell_0], \ldots, [\ell_0; \ell_0]\},$$

and $\mathbb{S}_- := \{[m; *]\}$. Pa-III implies that C3(a) is satisfied. Since $\mathbb{P}$ has code $\langle \cdots m \cdots 0 \cdots \rangle$, which is impossible when $A < 0$, C4 is automatically satisfied.

*Pa-IV:* Suppose $\mathbb{P}$ has code $\langle \ell_0 \, 0 \cdots (\ell_0 + 1) \cdots m \cdots \rangle$ for some $0 \leq \ell_0 \leq m - 2$. Take $y := [\ell_0 + 1; \ell_0]$, $\mathbb{S}_+ := \{[m; *]\}$, and $\mathbb{S}_- := \{[*; 0]\}$. We check C3. Given $\delta := [m; j] \in \mathbb{S}_+$, if $j \neq \ell_0$, then

$$\begin{aligned} \mathbb{P} \text{ has code } \langle \cdots m \cdots j \cdots \rangle \quad &\Rightarrow \quad \sigma_A(\delta) \geq 0, \\ \text{while } \mathbb{P} \text{ has code } \langle \ell_0 \, 0 \cdots j \cdots m \cdots \rangle \quad &\Rightarrow \quad 0 \geq \sigma_A(\delta) \geq \sigma_A([m; 0]) = \sigma_A(q - 1). \end{aligned}$$

Hence C3(a) or (b) is satisfied. Else if $j = \ell_0$, then for all $\ell \geq 1$, we have

$$\begin{aligned} \left(1 - \frac{1}{A^{\ell-1}} + \frac{\sigma_A(\delta)}{A^\ell}\right) - \sigma_A(y) &= \left(1 - \frac{1}{A^{\ell-1}} + \frac{\sigma_A([m; \ell_0])}{A^\ell}\right) - \sigma_A(\partial_{\ell_0+1}) + \sigma_A(\partial_{\ell_0}) \\ &= (1 + \sigma_A(\partial_{\ell_0})) \left(1 - \frac{1}{A^\ell}\right) - \sigma_A(\partial_{\ell_0+1}) \\ &\geq -\sigma_A(\partial_{\ell_0+1}) > 0. \end{aligned}$$

Therefore C3(d) is satisfied. Finally, we check C4. Given $\delta := [j; 0] \in \mathbb{S}_-$, if $j = \ell_0$ then $|\sigma_A(\delta)| < |A| \cdot |\sigma_A(y)|$, else if $j \neq \ell_0$ then $\sigma_A(\delta) \leq 0$. Thus C4(c) or (a) is satisfied.

**Q.E.D.**

We settle the real cases now.

**Proof of the real cases (p.3 Theorem 1.3)**

When $m = 1$ or 2, the last digit in the code of $\mathbb{P} = \mathbb{P}(A)$ can only be 0, 1, or $m$. Hence Pa-I in Theorem 1.5 (p.4) is matched and $|A| < 1$. When $m = 3$, all of the 24 possible codes of $\mathbb{P}$ are:

$$\begin{array}{llllll} \langle 0123 \rangle^{(I)} & \langle 0132 \rangle^{(I')} & \langle 0213 \rangle^{(I)} & \langle 0231 \rangle^{(I)} & \langle 0312 \rangle^{(I')} & \langle 0321 \rangle^{(I)} \\ \langle 1023 \rangle^{(I)} & \langle 1032 \rangle^{(II)} & \langle 1203 \rangle^{(I)} & \langle 1230 \rangle^{(I)} & \langle 1302 \rangle^{(II)} & \langle 1320 \rangle^{(I)} \\ \langle 2013 \rangle^{(I)} & \langle 2031 \rangle^{(I)} & \langle 2103 \rangle^{(I)} & \langle 2130 \rangle^{(I)} & \langle 2301 \rangle^{(I)} & \langle 2310 \rangle^{(I)} \\ \langle 3012 \rangle^{(I')} & \langle 3021 \rangle^{(I)} & \langle 3102 \rangle^{(I')} & \langle 3120 \rangle^{(I)} & \langle 3201 \rangle^{(I)} & \langle 3210 \rangle^{(I)} \end{array}$$

The superscripts indicate which pattern the code matches. Therefore $|A| < 1$ again.

Suppose $m = 4$ and $\mathbb{P}$ has code $\langle i_4 i_3 i_2 i_1 i_0 \rangle$. If on the contrary $|A| > 1$, then $\langle i_4 i_3 i_2 i_1 i_0 \rangle$ has to avoid all patterns in Theorem 1.5. Pa-I and Pa-I′ force $i_4 \in \{1, 2\}$ and $i_0 \in \{2, 3\}$:

| $i_4$ | $i_3 \quad i_2 \quad i_1$ | $i_0$ |
|---|---|---|
| $\times 4, 3, 0$ | | $\times 0, 1, 4$ |
| $1, 2$ | | $2, 3$ |

As we cannot have $(i_4, i_0) = (2, 2)$, and the cases $(i_4, i_0) = (2, 3)$ or $(1, 2)$ are Pa-II, it remains to consider the case $(i_4, i_0) = (1, 3)$. We want to avoid Pa-II and Pa-II′, and there are two cases: $A > 1$ or $A < -1$. In the former case, we need

$$\langle i_4 i_3 i_2 i_1 i_0 \rangle = \begin{cases} \langle \cdots 4 \cdots 0 \cdots \rangle \\ \langle 1 \cdots 2 \cdots 4 \cdots \rangle \\ \langle \cdots 0 \cdots 2 \cdots 3 \rangle , \end{cases}$$

which results in no valid code. In the latter case, we need

$$\langle i_4 i_3 i_2 i_1 i_0 \rangle = \begin{cases} \langle \cdots 0 \cdots 4 \cdots \rangle \\ \langle 1 \cdots 2 \cdots 4 \cdots \rangle \\ \langle \cdots 0 \cdots 2 \cdots 3 \rangle . \end{cases}$$

The only possibility is $\langle 10243 \rangle$, which matches Pa-IV however. This completes the proof.

**Q.E.D.**

**Proof of "125034" (p.4 Theorem 1.6)**

Suppose $A \in (1, q)$ and $\mathbb{P}$ has code $\langle i_5\, i_4\, i_3\, i_2\, i_1\, i_0 \rangle$. This code has to avoid all patterns in Theorem 1.5 (p.4). Pa-I and Pa-I′ force $i_5 \in \{1, 2, 3\}$ and $i_0 \in \{2, 3, 4\}$:

| $i_5$ | $i_4 \quad i_3 \quad i_2 \quad i_1$ | $i_0$ |
|---|---|---|
| $\times 5, 4, 0$ | | $\times 0, 1, 5$ |
| $1, 2, 3$ | | $2, 3, 4$ |

Consider $i_5 = 3$. As we cannot have $(i_5, i_0) = (3, 3)$, and the cases $(i_5, i_0) = (3, 4)$ is Pa-II, it follows that $(i_5, i_0) = (3, 2)$. By $A > 1$, Pa-II and Pa-II′, we have

$$\langle i_5\, i_4\, i_3\, i_2\, i_1\, i_0 \rangle = \begin{cases} \langle \cdots 5 \cdots 0 \cdots \rangle \\ \langle 3 \cdots 4 \cdots 5 \cdots \rangle \\ \langle \cdots 0 \cdots 1 \cdots 2 \rangle , \end{cases}$$

implying $\langle i_5\, i_4\, i_3\, i_2\, i_1\, i_0 \rangle = \langle 345012 \rangle$. But this matches Pa-III (with $\ell_0 = 2$).

Next we consider $i_5 = 2$. Pa-II forces $i_0 = 4$. By $A > 1$, Pa-II and Pa-II′, we require

$$\langle i_5\, i_4\, i_3\, i_2\, i_1\, i_0 \rangle = \begin{cases} \langle \cdots 5 \cdots 0 \cdots \rangle \\ \langle 2 \cdots 3 \cdots 5 \cdots \rangle \\ \langle \cdots 0 \cdots 3 \cdots 4 \rangle , \end{cases}$$

which does not give any valid code representation.

Finally, consider $i_5 = 1$. Pa-II forces $i_0 = 3, 4$. We have already handled the case $(i_5, i_0) = (2, 4)$, so by an appeal to symmetry (p.11 Observation 2.1), the case

$(i_5, i_0) = (1, 3)$ can be ignored, and so $i_0 = 4$. By $A > 1$, Pa-II and Pa-II′, we have

$$\langle i_5\, i_4\, i_3\, i_2\, i_1\, i_0\rangle = \begin{cases} \langle \cdots 5 \cdots 0 \cdots \rangle \\ \langle 1 \cdots 2 \cdots 5 \cdots \rangle \\ \langle \cdots 0 \cdots 3 \cdots 4 \rangle\,, \end{cases}$$

whence $\langle i_5\, i_4\, i_3\, i_2\, i_1\, i_0\rangle = \langle 125034\rangle$. This completes the proof.

**Q.E.D.**

**Proof of "120534" (p.4 Theorem 1.7)**

Suppose $A \in [-q, -1)$ and $\mathbb{P}$ has code $\langle i_5\, i_4\, i_3\, i_2\, i_1\, i_0\rangle$. This code has to avoid all patterns in Theorem 1.5 (p.4). Pa-I and Pa-I′ force $i_5 \in \{1, 2, 3\}$ and $i_0 \in \{2, 3, 4\}$:

| $i_5$ | $i_4$ $i_3$ $i_2$ $i_1$ | $i_0$ |
|---|---|---|
| $\times 5, 4, 0$ | | $\times 0, 1, 5$ |
| $1, 2, 3$ | | $2, 3, 4$ |

Consider $i_5 = 3$. Pa-II forces $(i_5, i_0) = (3, 2)$. By $A < -1$, Pa-II and Pa-II′, we require

$$\langle i_5\, i_4\, i_3\, i_2\, i_1\, i_0\rangle = \begin{cases} \langle \cdots 0 \cdots 5 \cdots \rangle \\ \langle 3 \cdots 4 \cdots 5 \cdots \rangle \\ \langle \cdots 0 \cdots 1 \cdots 2 \rangle\,. \end{cases}$$

If the second digit $i_4$ is 0, then the middle constraint implies the code matches Pa-IV. Similarly, if the second last digit $i_1$ is 5, then the code would match Pa-IV′. As a result, $\langle i_5\, i_4\, i_3\, i_2\, i_1\, i_0\rangle = \langle 340512\rangle$. But this contradicts p.13 Proposition 2.5.

Next, we consider $i_5 = 2$. Pa-II forces $i_0 = 4$. By $A < -1$, Pa-II and Pa-II′, we have

$$\langle i_5 i_4 i_3 i_2 i_1 i_0\rangle = \begin{cases} \langle \cdots 0 \cdots 5 \cdots \rangle \\ \langle 2 \cdots 3 \cdots 5 \cdots \rangle \\ \langle \cdots 0 \cdots 3 \cdots 4 \rangle\,. \end{cases}$$

Again the second digit $i_4$ cannot be 0, and the second last digit $i_1$ cannot be 5. These force $i_4 = 1 = i_1$, which is a contradiction.

Finally, consider $i_5 = 1$. Pa-II forces $i_0 = 3, 4$. The case $(i_5, i_0) = (1, 3)$ can be ignored by Observation 2.1 (p.11), so $i_0 = 4$. By $A < -1$, Pa-II and Pa-II′, we have

$$\langle i_5\, i_4\, i_3\, i_2\, i_1\, i_0\rangle = \begin{cases} \langle \cdots 0 \cdots 5 \cdots \rangle \\ \langle 1 \cdots 2 \cdots 5 \cdots \rangle \\ \langle \cdots 0 \cdots 3 \cdots 4 \rangle\,. \end{cases}$$

Again the second digit $i_4$ cannot be 0, and the second last digit $i_1$ cannot be 5, whence $\langle i_5\, i_4\, i_3\, i_2\, i_1\, i_0\rangle = \langle 120534\rangle$. This completes the proof.

**Q.E.D.**

## 3.2 Interlude: the secret of <125034>

The result of $\langle 125034\rangle$ (p.4 Theorem 1.6) is mysterious because it eliminates all but one pattern. How does an example corresponding to this chosen pattern, if any, really look like? In this section, we shall try to unmask the secret of $\langle 125034\rangle$.

**Proposition 3.1 (<125034>: Necessity of large gap)**
*Let $A$ be a real positive algebraic conjugate of $q$. Suppose $\Phi$ satisfies the FTC and $\mathbb{P} = \mathbb{P}(A)$ has code $\langle \cdots (\ell_0+1) \cdots m \cdots 0 \cdots \ell_0 \cdots (m-1) \rangle$, where $0 \le \ell_0 \le m-1$. If*

$$[\ell_0+1;\ell_0] \le \frac{q-1}{q} + \sum_{i=2}^{\infty} \frac{[m-1;0]}{q^i},$$

*then $A < 1$. In particular, if $[m;m-1] \le \dfrac{q(q-1)}{q(q-1)+1}$, then $A < 1$.*

**Proof**
The proof is similar to that of p.15 Proposition 2.6. Suppose on the contrary $A \in (1,q)$. Let $y := [\ell_0+1;\ell_0]$ and $\mathbb{T}_i^- := \mathbb{T}_i^-(A,1)$. By our hypothesis on $[\ell_0+1;\ell_0]$, we have $y \in (0,1)$, so we can let $k_0 = k_0(y)$ be the smallest non-negative integer such that

$$y \le \sum_{i=1}^{k_0} \frac{q-1}{q^i} + \sum_{i=k_0+1}^{\infty} \frac{\max \mathbb{T}_i^-}{q^i} = \sum_{i=1}^{k_0} \frac{q-1}{q^i} + \sum_{i=k_0+1}^{\infty} \frac{[m-1;0]}{q^i}.$$

Note that by our hypothesis on $[\ell_0+1;\ell_0]$, $k_0$ can only be 0 or 1. To be parallel with the proof of Proposition 2.6, we still define

$$k_0^* = \begin{cases} 1 & \text{if } k_0 = 0 \\ k_0 & \text{otherwise} \end{cases}$$

(although it is always equal to 1 now).

We now define for each $i \ge k_0^*$ a set $\mathbb{E}_i \subseteq (\mathsf{D}(q) - \mathsf{D}(q))$. If $k_0 = 0$, then we define $\mathbb{E}_i := \mathbb{T}_i^-$ for all $i \ge k_0^*$. Else if $k_0 = 1$, then we define $\mathbb{E}_i$ by

$$\mathbb{E}_i := \begin{cases} \mathbb{T}_i^- \cup \{q-1\} & \text{if } i = k_0^* \\ \mathbb{T}_i^- & \text{if } i > k_0^*. \end{cases}$$

With these $\mathbb{E}_i$, we have

$$\sum_{i=k_0^*}^{\infty} \frac{\min \mathbb{E}_i}{q^i} \le 0 < y = y - \sum_{i=1}^{k_0^*-1} \frac{q-1}{q^i} \le \sum_{i=k_0^*}^{\infty} \frac{\max \mathbb{E}_i}{q^i}.$$

Observe that for any $j \ge k_0^*$, we have

$$\operatorname{gap}_{\max}(\mathbb{E}_j) \le 1 = q^j \sum_{i=j+1}^{\infty} \frac{\max \mathbb{E}_i - \min \mathbb{E}_i}{q^i},$$

because $[m;0] - [m-1;0] = [m;m-1] \le 1$. Therefore, applying our lazy algorithm (p.12 Proposition 2.3) to $x := y$, we obtain

$$y = \sum_{i=k_0^*}^{\infty} \frac{t_i}{q^i}$$

with $t_i \in \mathbb{E}_i$ for all $i \geq k_0^*$. As a consequence of the FTC (p.11 Proposition 2.2), we have

$$\sigma_A(y) = \sum_{i=k_0^*}^{\infty} \frac{\sigma_A(t_i)}{A^i}. \tag{3.1}$$

We now show that this is a contradiction. Using the code representation of $\mathbb{P}$, we have $\sigma_A(y) = \sigma_A([\ell_0+1;\ell_0]) \geq \sigma_A([m;0]) = A-1 > 0$. On the other hand, as $-\mathbb{T}_i^- \subseteq \mathbb{L}_i(A,1)$, so if $k_0 = 0$, then by (3.1)

$$0 < \sigma_A(y) = \sum_{i=1}^{\infty} \frac{\sigma_A(t_i)}{A^i} \leq 0,$$

giving a contradiction. Else if $k_0 = 1$, (3.1) gives

$$0 < A-1 \leq \sigma_A(y) = \frac{\sigma_A(t_1)}{A} + \sum_{i=2}^{\infty} \frac{\sigma_A(t_i)}{A^i} \leq \frac{\sigma_A(t_1)}{A}.$$

If $t_1 \in \mathbb{T}_1^-$, then the R.H.S. is $\leq 0$, which gives a contradiction. As a result, we have $t_1 = q-1$, and the above gives

$$0 < A-1 \leq \frac{\sigma_A(q-1)}{A} = \frac{A-1}{A},$$

which is also a contradiction.

Finally, we check what the hypothesis on $[\ell_0+1;\ell_0]$ means when $\ell_0 = m-1$. Now

$$[m;m-1] \leq \frac{q-1}{q} + \sum_{i=2}^{\infty} \frac{[m-1;0]}{q^i} = 1 - \frac{1}{q} + \frac{[m-1;0]}{q(q-1)} = 1 - \frac{[m;m-1]}{q(q-1)},$$

whence

$$[m;m-1] \leq \frac{q(q-1)}{q(q-1)+1}.$$

**Q.E.D.**

The preceding proposition, applied to $\langle 125034 \rangle$, suggests that we need the gap $[5;4]$ to be large. By symmetry, we may need $[1;0]$ to be large too. A natural response is to consider the case $[5;4] = [1;0] = 1$. It turns out to be productive.

**<u>Proposition 3.2 (<125034>: Towards the first example)</u>**

*Let $m = 5$ and $A$ be a real positive algebraic conjugate of $q$. Suppose $\Phi$ satisfies the FTC, $[1;0] = [5;4] = 1$, $A > 1$, and $\mathbb{P} = \mathbb{P}(A)$ has code $\langle 125034 \rangle$. Then $A^2 - 6A + 6 \geq 0$. In case $A^2 - 6A + 6 = 0$, we have $A = 3 - \sqrt{3}$, $q = 3 + \sqrt{3}$, and*

| $\partial_1$ | $\partial_2$ | $\partial_5$ | $\partial_0$ | $\partial_3$ | $\partial_4$ |
|---|---|---|---|---|---|
| $1$ | $q/3$ | $q-1$ | $0$ | $2q/3-1$ | $q-2$ |
| $1$ | $1.577$ | $3.732$ | $0$ | $2.154$ | $2.732$ |

.

**Proof**

Firstly, we claim that for all $y \in \{[2;1], [3;2], [4;3]\}$, we have

$$|\sigma_A(y)| \leq \frac{|\sigma_A([5;1])|}{A}.$$

To verify the claim, suppose on the contrary the above does not hold for some $y \in \{[2;1], [3;2], [4;3]\}$. Let $\mathfrak{w} := 1$ and $\mathbb{S}_\ell := \{[5;*]\}$ for all $\ell \geq 1$. We apply p.15 Proposition 2.6 using these $y$, $\mathfrak{w}$, and $\mathbb{S}_\ell$. For condition 3(c) of that proposition, note that if 3(c)i does not hold, then $\delta = [5;1]$ or $[5;2]$, and our assumption on $y$ implies that 3(c)iii holds. Accordingly, we have $|A| \leq 1$, which is a contradiction.

Therefore, by the hypothesis $[1;0] = [5;4] = 1$, we are considering the IFS

| $\sigma_A(\partial_1)$ | $\sigma_A(\partial_2)$ | $\sigma_A(\partial_5)$ | $\sigma_A(\partial_0)$ | $\sigma_A(\partial_3)$ | $\sigma_A(\partial_4)$ |
|---|---|---|---|---|---|
| $1$ | $a$ | $A-1$ | $0$ | $b$ | $A-2$ |

for some $a, b \in \mathbb{R}$ satisfying

$$\begin{cases} 1 - a \leq (2-A)/A \\ a - b \leq (2-A)/A \\ b - (A-2) \leq (2-A)/A. \end{cases} \tag{3.2}$$

Adding the inequalities, we get

$$3 - A \leq \frac{6-3A}{A} \quad \Rightarrow \quad 0 \leq A^2 - 6A + 6.$$

When $A^2 - 6A + 6 = 0$, by $|A| \leq q$ we have $A = 3 - \sqrt{3}$ and $q = 3 + \sqrt{3}$. Also, equality holds in (3.2). As $\dfrac{A(A-6)}{-6} = 1$, we have $\dfrac{1}{A} = \dfrac{6-A}{6}$. It follows from the first and third equation of (3.2) that

$$\begin{cases} a = 2 - \dfrac{2}{A} = 2 - 2 \cdot \dfrac{6-A}{6} = \dfrac{A}{3} \\ b = A - 3 + \dfrac{2}{A} = A - 3 + 2 \cdot \dfrac{6-A}{6} = \dfrac{2A}{3} - 1. \end{cases}$$

**Q.E.D.**

As $3 - \sqrt{3} \approx 1.2679492$, we see that $q := 3 + \sqrt{3} \approx 4.7320508$ is not a PV number. The IFS discovered in the preceding proposition,

$$\Phi := \{\rho x + \partial_i \rho\}_{i=0}^{5},$$

where

$$\begin{aligned} \{\partial_i\}_{i=0}^{5} :=& \left\{ \quad 0, \quad 1, \quad \frac{q}{3}, \quad \left(\frac{2q}{3} - 1\right), \quad (q-2), \quad (q-1) \quad \right\} \\ =& \left\{ \quad 0, \quad 1, \quad \left(1 + \frac{1}{\sqrt{3}}\right), \quad \left(1 + \frac{2}{\sqrt{3}}\right), \quad (1+\sqrt{3}), \quad (2+\sqrt{3}) \quad \right\} \\ =& \left\{ \quad 0, \quad 1, \quad 1.5773503, \quad 2.1547005, \quad 2.7320508, \quad 3.7320508 \quad \right\}, \end{aligned} \tag{3.3}$$

is also of the required form of Chapter 2 section 2.1. And finally, *yes*. This IFS indeed satisfies the FTC. Bravo! Let's … defer the verification until Chapter 5, nevertheless, for we can give a more systematic treatment of the issue there. We ask the reader's pardon for the deferment.

We end by describing an implication of the FTC for this $\Phi$. Now, as a consequence of the FTC, $Y_{\text{core}}$ is a finite set. We shall verify in Chapter 5 that $Y_{\text{core}} = \{1, q/3-1\}$. Using the notation in p.16 Lemma 2.7, we define

$$X_n := q^n F_n = \left\{ \sum_{i=0}^{n-1} q^i d_i : \ d_i \in \mathsf{D}(q) \right\}.$$

By that lemma, we have $\left\{ x - x^{(\downarrow X_n)} : \ x \in X_n^{\times \min} \right\} \subseteq Y_{\text{core}}$. Let

$$\begin{aligned} N_1(n) &:= \# \left\{ x \in X_n : x - x^{(\downarrow X_n)} = 1 \right\}, \\ N_2(n) &:= \# \left\{ x \in X_n : x - x^{(\downarrow X_n)} = q/3 - 1 \right\}. \end{aligned}$$

It follows that $\# X_n = N_1(n) + N_2(n) + 1$ and

$$N_1(n) \cdot 1 + N_2(n) \cdot \left( \frac{q}{3} - 1 \right) = \sum_{x \in X_n^{\times \min}} (x - x^{\downarrow}) = q^n - 1.$$

The last equation gives

$$\begin{pmatrix} 1 & \dfrac{q}{3} - 1 \\ 1 & \dfrac{A}{3} - 1 \end{pmatrix} \begin{pmatrix} N_1(n) \\ N_2(n) \end{pmatrix} = \begin{pmatrix} q^n - 1 \\ A^n - 1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1 & \dfrac{q}{3} - 1 \\ 1 & \dfrac{A}{3} - 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \dfrac{\sqrt{3}}{3} \\ 1 & \dfrac{-\sqrt{3}}{3} \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{\sqrt{3}}{2} & \dfrac{-\sqrt{3}}{2} \end{pmatrix},$$

we have

$$\begin{cases} N_1(n) = \dfrac{q^n + A^n}{2} - 1 \\ N_2(n) = \dfrac{\sqrt{3}}{2}(q^n - A^n) \\ \# X_n = \dfrac{\sqrt{3}+1}{2} q^n - \dfrac{\sqrt{3}-1}{2} A^n. \end{cases} \tag{3.4}$$

As $q$, $A$ are the roots of $x^2 - 6x + 6$, recalling the theory of linear recurrence relation, we also have

$$\begin{cases} N_1(n+2) + 1 = 6[N_1(n+1) + 1] - 6[N_1(n) + 1] = 6N_1(n+1) - 6N_1(n) \\ N_2(n+2) = 6N_2(n+1) - 6N_2(n) \\ \# X_{n+2} = 6\left(\# X_{n+1}\right) - 6\left(\# X_n\right). \end{cases} \tag{3.5}$$

For instance, taking $n = 1$ in (3.4), we have $N_1(n) = 2$, $N_2(n) = 3$, and $\#X_n = 6$. This matches the data in (3.3). Taking $n = 2$ on the other hand, we have $N_1(n) = 11$, $N_2(n) = 18$, and $\#X_n = 30$. This matches the structure of $X_2$, as the following tables show (we use the shorthand $\langle i\rangle := \partial_i$ in the meantime).

| $x \in X_2$ | value | $x - x^{\downarrow}$ |
|---|---|---|
| $\langle 0\rangle q + \langle 0\rangle$ | 0 | $\infty$ |
| $\langle 0\rangle q + \langle 1\rangle$ | 1 | 1 |
| $\langle 0\rangle q + \langle 2\rangle$ | 1.577 | $q/3 - 1$ |
| $\langle 0\rangle q + \langle 3\rangle$ | 2.154 | $q/3 - 1$ |
| $\langle 0\rangle q + \langle 4\rangle$ | 2.732 | $q/3 - 1$ |
| $\langle 0\rangle q + \langle 5\rangle$ | 3.732 | 1 |

| $x \in X_2$ | value | $x - x^{\downarrow}$ |
|---|---|---|
| $\langle 1\rangle q + \langle 0\rangle$ | 4.732 | 1 |
| $\langle 1\rangle q + \langle 1\rangle$ | 5.732 | 1 |
| $\langle 1\rangle q + \langle 2\rangle$ | 6.309 | $q/3 - 1$ |
| $\langle 1\rangle q + \langle 3\rangle$ | 6.886 | $q/3 - 1$ |
| $\langle 2\rangle q + \langle 0\rangle = \langle 1\rangle q + \langle 4\rangle$ | 7.464 | $q/3 - 1$ |
| $\langle 2\rangle q + \langle 1\rangle = \langle 1\rangle q + \langle 5\rangle$ | 8.464 | 1 |

| $x \in X_2$ | value | $x - x^{\downarrow}$ |
|---|---|---|
| $\langle 2\rangle q + \langle 2\rangle$ | 9.041 | $q/3 - 1$ |
| $\langle 2\rangle q + \langle 3\rangle$ | 9.618 | $q/3 - 1$ |
| $\langle 3\rangle q + \langle 0\rangle = \langle 2\rangle q + \langle 4\rangle$ | 10.196 | $q/3 - 1$ |
| $\langle 3\rangle q + \langle 1\rangle = \langle 2\rangle q + \langle 5\rangle$ | 11.196 | 1 |
| $\langle 3\rangle q + \langle 2\rangle$ | 11.773 | $q/3 - 1$ |
| $\langle 3\rangle q + \langle 3\rangle$ | 12.350 | $q/3 - 1$ |

| $x \in X_2$ | value | $x - x^{\downarrow}$ |
|---|---|---|
| $\langle 4\rangle q + \langle 0\rangle = \langle 3\rangle q + \langle 4\rangle$ | 12.928 | $q/3 - 1$ |
| $\langle 4\rangle q + \langle 1\rangle = \langle 3\rangle q + \langle 5\rangle$ | 13.928 | 1 |
| $\langle 4\rangle q + \langle 2\rangle$ | 14.505 | $q/3 - 1$ |
| $\langle 4\rangle q + \langle 3\rangle$ | 15.082 | $q/3 - 1$ |
| $\langle 4\rangle q + \langle 4\rangle$ | 15.660 | $q/3 - 1$ |
| $\langle 4\rangle q + \langle 5\rangle$ | 16.660 | 1 |

| $x \in X_2$ | value | $x - x^{\downarrow}$ |
|---|---|---|
| $\langle 5\rangle q + \langle 0\rangle$ | 17.660 | 1 |
| $\langle 5\rangle q + \langle 1\rangle$ | 18.660 | 1 |
| $\langle 5\rangle q + \langle 2\rangle$ | 19.237 | $q/3 - 1$ |
| $\langle 5\rangle q + \langle 3\rangle$ | 19.814 | $q/3 - 1$ |
| $\langle 5\rangle q + \langle 4\rangle$ | 20.392 | $q/3 - 1$ |
| $\langle 5\rangle q + \langle 5\rangle$ | 21.392 | 1 |

Finally, using (3.5), we have $N_1(n) = 53$, $N_2(n) = 90$, and $\#X_n = 144$ when $n = 3$. These numbers can be demonstrated with the aid of computer.

# Chapter 4

# General cases

## 4.1 Conjugates on the unit circle

We start to deal with complex algebraic conjugates in this chapter. To begin with, note that if $|A| = 1$, which may happen now, then we cannot apply p.11 Proposition 2.2(c). This section is devoted to establishing some tools for such case.

Our approach is inspired by [1, proof of proposition 3.2]. Unlike the original argument, we simply consider algebraic conjugates of $q$ rather than general complex numbers on the unit circle.

We make an observation which was suggested by Feng (personal communication, 2021). Given an algebraic conjugate $A$ of $q$ with $|A| = 1$, it only happens that $(\arg A)/\pi \notin \mathbb{Q}$, for otherwise there exists $n \geq 1$ such that $A^n = 1$, which leads to the contradiction $q^n = 1$. Accordingly, we have the following lemma, which corresponds to [1, Lemma 3.6].

**Lemma 4.1 (Outcome of irrational rotation)**
*Let $A$ be an algebraic conjugate of $q$ and $w_0 \in \mathbb{C} \setminus \{0\}$ a nonzero complex number. If $|A| = 1$, then given any $z \in \mathbb{C} \setminus \{0\}$, we have $\Re\left(zw_0/A^i\right) = 0$ for at most one $i \in \mathbb{N}$.*

**Proof**
Since $(\arg A)/\pi \notin \mathbb{Q}$, we have

$$\begin{cases} \Re\left(zw_0/A^i\right) = 0 \\ \Re\left(zw_0/A^j\right) = 0 \end{cases} \Rightarrow \arg\left(zw_0/A^j\right) - \arg\left(zw_0/A^i\right) \in \mathbb{Z}\pi \Rightarrow i = j.$$

**Q.E.D.**

The next result shows that, given a sequence $\{\sigma_n\}_{n=1}^{\infty}$ of complex numbers with positive and increasing real parts, if the set $\{|\sigma_n - \sigma_{n-1}| : n \geq 2\}$ of distances between consecutive terms is a finite set, then the sequence cannot lie on a finite number of circles centred at the origin.

**Proposition 4.2 (Dilemma of finiteness)**
*Let $\{\sigma_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. Let $\Delta_n := \sigma_n - \sigma_{n-1}$. Suppose the following hold:*

(i) $\{|\Delta_n|\}_{n=2}^{\infty}$ *is a finite set;*

(ii) $\Delta_n \neq 0$ *infinitely often;*

(iii) *Either* $\Delta_n = 0$ *or* $\Re(\Delta_n) > 0$*;*

(iv) $\Re(\sigma_1) > 0$.

*Then* $\{|\sigma_n|\}_{n=1}^{\infty}$ *cannot be a finite set.*

**Proof**

It is proved by contradiction. Suppose on the contrary $\{|\sigma_n|\}_{n=1}^{\infty}$ is a finite set. A fortiori $\{\Re(\sigma_n)\}_{n=1}^{\infty}$ is bounded. As $\{\Re(\sigma_n)\}_{n=1}^{\infty}$ is an increasing sequence, it converges and hence

$$\Re(\Delta_i) = \Re(\sigma_i) - \Re(\sigma_{i-1}) \to 0 \quad \text{as } i \to \infty. \tag{4.1}$$

Let $\mathcal{C} := \left\{R \in \{|\sigma_n|\}_{n\geq 1} : |\sigma_i| = R \text{ for infinitely many } i\right\}$. Since $\{|\sigma_n|\}_{n\geq 1}$ is a finite set, we can define $R_0 := \max \mathcal{C}$. There exists a subsequence $\left\{\sigma_{r_j}\right\}_{j=1}^{\infty}$, $r_1 > 1$, such that

1. $\left|\sigma_{r_j}\right| = R_0$ and $\Delta_{r_j} \neq 0$ for all $j$ (*by* $R_0 \in \mathcal{C}$ *and* $\Delta_i \neq 0$ *infinitely often*)
2. There is a $d_0 > 0$ such that $\left|\Delta_{r_j}\right| = d_0$ for all $j$ (*by the finiteness of* $\{|\Delta_n|\}_{n\geq 2}$)
3. $\Im\left(\sigma_{r_j}\right)$ are all of the same $\pm$ sign: $\left\{\Im\left(\sigma_{r_j}\right)\right\}_{j=1}^{\infty} \subseteq (0,\infty)$ or $\subseteq (-\infty, 0)$
   (*suppose* $\left\{\sigma_{r_j}\right\}$ *is a subsequence satisfying requirement 1-2. If* $\Im\left(\sigma_{r_j}\right) = 0$ *for some* $j$*, then by* $\Re(\sigma_1) > 0$ *and* $\Re(\Delta_i) \geq 0$ *for all* $i$*, we see that writing* $\sigma_{r_j} = \sigma_{1+r_j} = \sigma_{2+r_j} = \cdots = \sigma_{\ell+r_j} \neq \sigma_{\ell+1+r_j}$*, we have* $\left|\sigma_{r_j}\right| < \left|\sigma_{\ell+1+r_j}\right|$*. Hence by the definition of* $R_0$*, we can take a subsequence of* $\left\{\sigma_{r_j}\right\}$ *so that this "same* $\pm$ *sign" property is satisfied.*)

   Define $u := 1$ if $\left\{\Im\left(\sigma_{r_j}\right)\right\}_{j=1}^{\infty} \subseteq (0,\infty)$, and $u := -1$ otherwise.
4. There exists $R_1 \in \mathcal{C}$ such that $\left|\sigma_{-1+r_j}\right| = R_1$ for all $j$ (*by finiteness of* $\{|\sigma_n|\}_{n\geq 1}$)
5. $\left\{\sigma_{r_j}\right\}$ converges: $\Re\left(\sigma_{r_j}\right) \uparrow a_0 > 0, \Im\left(\sigma_{r_j}\right) \to b_0$ (*by the compactness of the circle* $|z| = R_0$)
6. $\left|\Im\left(\Delta_{r_j}\right)\right| \to d_0$ (*by requirement 2 and* (4.1)), whence $b_0 \neq 0$
   (*we have*

$$\begin{aligned}
&(\Re\left(\sigma_{r_j}\right) - \Re\left(\Delta_{r_j}\right))^2 + (\Im\left(\sigma_{r_j}\right) - \Im\left(\Delta_{r_j}\right))^2 = R_1^2 \leq R_0^2 \\
&\Rightarrow R_0^2 + \mathcal{O}\big(\Re\left(\Delta_{r_j}\right)\big) - 2\Im\left(\sigma_{r_j}\right)\Im\left(\Delta_{r_j}\right) + \Im\left(\Delta_{r_j}\right)^2 \leq R_0^2 \\
&\Rightarrow \mathcal{O}\big(\Re\left(\Delta_{r_j}\right)\big) + \Im\left(\Delta_{r_j}\right)\left(\Im\left(\Delta_{r_j}\right) - 2\Im\left(\sigma_{r_j}\right)\right) \leq 0. \qquad (4.2)
\end{aligned}$$

   *If* $b_0 = 0$*, then letting* $j \to \infty$ *we have* $d_0^2 \leq 0$*, a contradiction.*)
7. $\Im\left(\Delta_{r_j}\right)$ are of the same $\pm$ sign as $u$ for all $j$, and $\Im\left(\Delta_{r_j}\right) \to ud_0$
   (*suppose* $\left\{\sigma_{r_j}\right\}$ *is a subsequence satisfying requirement 1-6. By* (4.2)*, given* $\varepsilon > 0$*, for all large* $j$ *we have* $\varepsilon > d_0^2 - 2\Im\left(\Delta_{r_j}\right) b_0 = d_0^2 - 2\Im\left(\Delta_{r_j}\right) u\,|b_0|$*. If* $u = 1$ *and* $\Im\left(\Delta_{r_j}\right) \leq 0$*, then this inequality gives* $\varepsilon > d_0^2$*, which is a contradiction for small* $\varepsilon$*. The same happens if* $u = -1$ *and* $\Im\left(\Delta_{r_j}\right) \geq 0$*. Therefore,* $\Im\left(\Delta_{r_j}\right)$ *is of the same* $\pm$ *sign as* $u$ *for all large* $j$*.*)

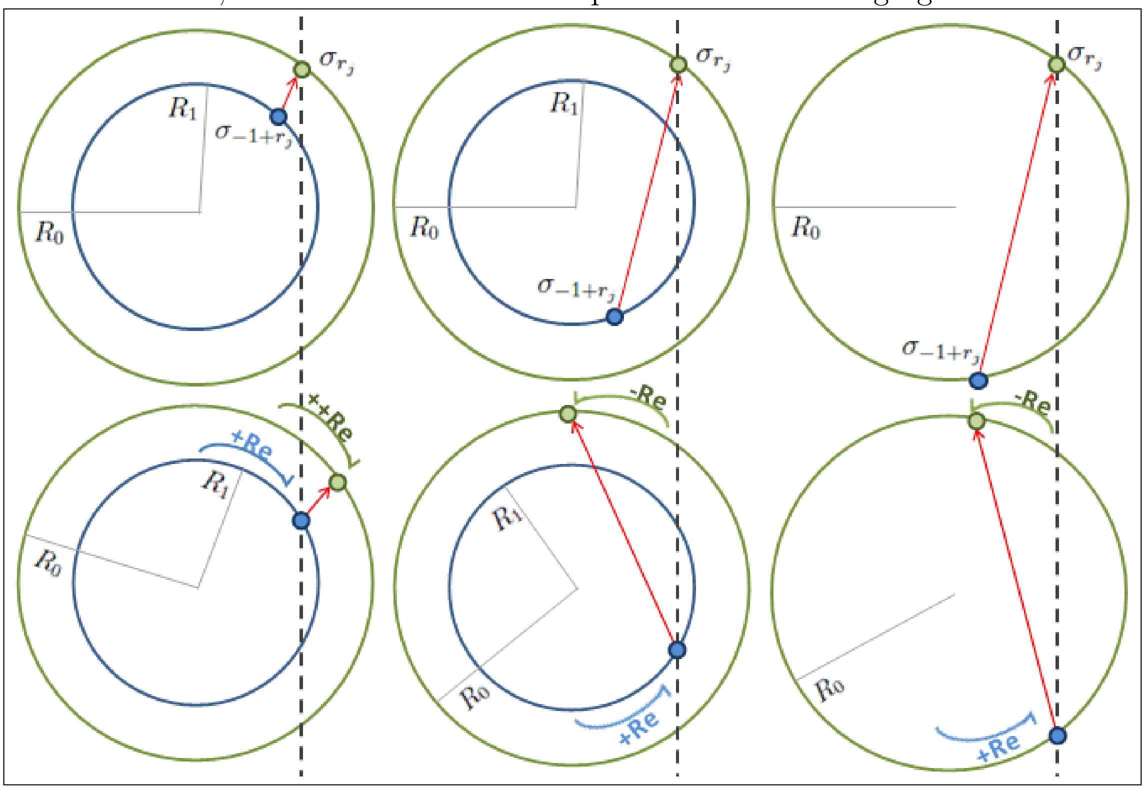
8. $\left\{\sigma_{-1+r_j}\right\}$ converges (by the compactness of the circle $|z| = R_1$)

9. Writing
$$\sigma_{r_j} := R_0 e^{i\phi_j}, \quad \sigma_{-1+r_j} := R_1 e^{i\theta_j}$$
where $\phi_j, \theta_j \in [0, 2\pi)$, there exists $\Gamma \in \mathbb{R}$ such that $\phi_j - \theta_j \equiv \Gamma$ for all $j$.
(*we have*
$$\begin{aligned}
&\left|\sigma_{r_j} - \sigma_{-1+r_j}\right|^2 = \left|\Delta_{r_j}\right|^2 \equiv d_0^2 = \textit{constant} \\
\Rightarrow\ &(R_0 \cos\phi_j - R_1 \cos\theta_j)^2 + (R_0 \sin\phi_j - R_1 \sin\theta_j)^2 \equiv \textit{constant} \\
\Rightarrow\ &\cos\phi_j \cos\theta_j + \sin\phi_j \sin\theta_j \equiv \textit{constant} \\
\Rightarrow\ &\cos(\phi_j - \theta_j) \equiv \textit{constant})
\end{aligned}$$

10. $\Im\left(\sigma_{-1+r_j}\right)$ are all of the same $\pm$ sign: $\left\{\Im\left(\sigma_{-1+r_j}\right)\right\}_{j=1}^{\infty} \subseteq (0, \infty)$ or $\subseteq (-\infty, 0)$
(*suppose* $\left\{\sigma_{r_j}\right\}$ *is a subsequence satisfying requirement 1-9. If* $\Im\left(\sigma_{-1+r_j}\right) = 0$ *for some* $j$*, then by* $\Re\left(\sigma_{-1+r_{j+1}}\right) \geq \Re\left(\sigma_{r_j}\right) > \Re\left(\sigma_{-1+r_j}\right) > 0$ *and* $\left|\Im\left(\sigma_{-1+r_{j+1}}\right)\right| \geq 0 = \left|\Im\left(\sigma_{-1+r_j}\right)\right|$*, we have* $R_1 = \left|\sigma_{-1+r_{j+1}}\right| > \left|\sigma_{-1+r_j}\right| = R_1$*, a contradiction. Therefore, by taking a subsequence of* $\left\{\sigma_{r_j}\right\}$ *if necessary, we can have this "same* $\pm$ *sign" property.)*
Define $v := 1$ if $\left\{\Im\left(\sigma_{-1+r_j}\right)\right\}_{j=1}^{\infty} \subseteq (0, \infty)$, and $v := -1$ otherwise.

This ends our construction of $\left\{\sigma_{r_j}\right\}_{j=1}^{\infty}$.

When $u = 1$, the situation is like the top row of the following figure:




The bottom row on the other hand illustrates why there should be contradictions. To be precise, write
$$R_0 e^{i\phi_*} := \lim_{j\to\infty} \sigma_{r_j}, \quad R_1 e^{i\theta_*} := \lim_{j\to\infty} \sigma_{-1+r_j},$$

where $\phi_*, \theta_* \in [0, 2\pi)$. As $\lim_{j\to\infty} \sigma_{-1+r_j} = \lim \sigma_{r_j} - \lim \Delta_{r_j} = a_0 + i(b_0 - ud_0)$, we have

$$\Re\left(R_0 e^{i\phi_*}\right) = a_0 = \Re\left(R_1 e^{i\theta_*}\right). \tag{4.3}$$

Also, observe that

$$\begin{aligned} \sigma_{-1+r_{j+1}} &= R_1 e^{i\theta_{j+1}} = e^{i(\theta_{j+1}-\theta_j)}\sigma_{-1+r_j}, \\ \sigma_{r_{j+1}} &= R_0 e^{i(\theta_{j+1}+\Gamma)} = e^{i(\theta_{j+1}-\theta_j)}\sigma_{r_j}. \end{aligned} \tag{4.4}$$

We shall finish the proof by showing a contradiction for each of the following four cases:

(i) $b_0 > 0$, $v = -1$;

(ii) $b_0 < 0$, $v = 1$;

(iii) $b_0 > 0$, $v = 1$;

(iv) $b_0 < 0$, $v = -1$.

Suppose case(i) happens. Then $\phi_* \in (0, \pi/2)$, and $3\pi/2 < \theta_1 < \theta_2 < \cdots < 2\pi$. There exists $\varepsilon_0 > 0$ such that $\phi_* + 2\varepsilon_0 < \pi/2$. With this $\varepsilon_0$, for all large $j$ we have

$$\begin{cases} 0 < \phi_j < \phi_* + \varepsilon_0 \\ 0 < \theta_{j+1} - \theta_j < \varepsilon_0, \end{cases}$$

whence $0 < \phi_j < \phi_j + \theta_{j+1} - \theta_j < \phi_* + 2\varepsilon_0 < \pi/2$. As a result, we have $0 < \cos(\theta_{j+1} - \theta_j + \phi_j) < \cos\phi_j$. But by (4.4), $\phi_{j+1} = \theta_{j+1} - \theta_j + \phi_j$, whence $\Re\left(\sigma_{r_{j+1}}\right) = R_0\cos\phi_{j+1} < R_0\cos\phi_j = \Re\left(\sigma_{r_j}\right)$, showing a contradiction. This means that case(i) is impossible. Case(ii) can be excluded by a similar argument.

Next, suppose case(iii) happens. Then $\phi_* \in (0, \pi/2)$, $\theta_* \in [0, \pi/2)$, $\theta_j \in (0, \pi/2)$, and $\theta_j \downarrow \theta_*$. By (4.4), we see that $\sigma_{-1+r_j} = e^{i(\theta_j - \theta_1)}\sigma_{-1+r_1}$ and $\sigma_{r_j} = e^{i(\theta_j-\theta_1)}\sigma_{r_1}$, so letting $j \to \infty$, we have

$$R_1 e^{i\theta_*} = e^{i(\theta_*-\theta_1)}\sigma_{-1+r_1}, \qquad R_0 e^{i\phi_*} = e^{i(\theta_*-\theta_1)}\sigma_{r_1}. \tag{4.5}$$

It follows that

$$\Re\left(R_1 e^{i\theta_*}\right) = \Re\left(e^{i(\theta_*-\theta_1)}\sigma_{-1+r_1}\right) = |\cos(\theta_*-\theta_1)|\Re\left(\sigma_{-1+r_1}\right) + |\sin(\theta_*-\theta_1)|\Im\left(\sigma_{-1+r_1}\right).$$

Since $b_0 > 0$, we have $u = 1$ and so $\Im\left(\Delta_{r_1}\right) > 0$. Together with $\Re\left(\Delta_{r_1}\right) > 0$, we get

$$\Re\left(R_1 e^{i\theta_*}\right) < |\cos(\theta_* - \theta_1)|\,\Re\left(\sigma_{r_1}\right) + |\sin(\theta_* - \theta_1)|\,\Im\left(\sigma_{r_1}\right) = \Re\left(R_0 e^{i\phi_*}\right) \quad \text{by (4.5)}.$$

This contradicts (4.3) however. This means that case(iii) is impossible. By a similar argument case(iv) can be excluded too. This completes the proof.

**Q.E.D.**

Below is our major tool for handling conjugates on the unit circle. The proof is adapted from [1, proof of Proposition 3.2].

**Lemma 4.3 (Sufficient condition for $|A| \neq 1$)**
*Let $A$ be an algebraic conjugate of $q$. Suppose there exist $y \in Y \setminus \{0\}$ and $\{s_i(q)\}_{i=1}^{\infty} \subseteq (\mathsf{D}(q) - \mathsf{D}(q))$ such that*

1. $y = \displaystyle\sum_{i=1}^{\infty} \frac{s_i(q)}{q^i}$;
2. *There exist $K_0 \geq 0$ and $\mathfrak{w} \in \mathbb{C} \setminus \{0\}$ such that*
$$C_+ := -\Re\left(\sigma_A(y)\mathfrak{w}\right) + \sum_{i=1}^{K_0} \Re\left(\frac{s_i(A)\mathfrak{w}}{A^i}\right) > 0$$
*and $\Re\left(\dfrac{s_i(A)\mathfrak{w}}{A^i}\right) \geq 0$ for all $i \geq K_0 + 1$.*

*If $\Phi$ satisfies the FTC, then $|A| \neq 1$.*

**Proof**
It is proved by contradiction. Suppose on the contrary $|A| = 1$. Since $y \in Y$, we can write

$$y =: -\sum_{i=-u_0}^{0} \frac{s_i(q)}{q^i}, \text{ where } s_i(q) \in (\mathsf{D}(q) - \mathsf{D}(q)) \text{ for all } -u_0 \leq i \leq 0.$$

Consider the sequence $\{s_i(q)\}_{i=-u_0}^{\infty}$. As $y \neq 0$, we have $s_i(q) \neq 0$ for some $-u_0 \leq i \leq 0$. We claim that $s_i(q) \neq 0$ infinitely often. For, suppose $s_i(q) = 0$ for all but a finite number of $i$. Letting $s_{p_0}(q)$ be the last non-zero term, by hypothesis we have

$$0 = \sum_{-u_0}^{\infty} \frac{s_i(q)}{q^i} = \sum_{-u_0}^{p_0} \frac{s_i(q)}{q^i}. \tag{4.6}$$

But then

$$0 = \Re\left(\mathfrak{w} \cdot \sigma_A\left(\sum_{-u_0}^{p_0} \frac{s_i(q)}{q^i}\right)\right) = \sum_{-u_0}^{p_0} \Re\left(\frac{s_i(A)\mathfrak{w}}{A^i}\right)$$
$$= \sum_{-u_0}^{K_0} \Re\left(\frac{s_i(A)\mathfrak{w}}{A^i}\right) + \sum_{K_0+1}^{\infty} \Re\left(\frac{s_i(A)\mathfrak{w}}{A^i}\right) = C_+ + \sum_{K_0+1}^{\infty} \Re\left(\frac{s_i(A)\mathfrak{w}}{A^i}\right) \geq C_+ > 0,$$

showing a contradiction. This verifies our claim.

Now, since $A^i$ gives rise to irrational rotation and $(\mathsf{D}(A) - \mathsf{D}(A))$ is a finite set, there exists a positive integer $N > K_0$ such that $\Re\left(s\mathfrak{w}/A^i\right) \neq 0$ for all nonzero $s \in (\mathsf{D}(A) - \mathsf{D}(A)) \setminus \{0\}$ and all $i \geq N$ (Lemma 4.1). Using such $N$, we define

$$\sigma_n := \sum_{i=-u_0}^{n+N} \frac{s_i(A)\mathfrak{w}}{A^i}.$$

As $\Re\left(\sigma_1\right) \geq C_+$, we are in the same situation as in the preceding Proposition 4.2. Therefore, $\{|\sigma_n|\}_{n=1}^{\infty}$ cannot be a finite set. However, this contradicts the FTC (p.11 Proposition 2.2). This completes the proof.

**Q.E.D.**

## 4.2 Proof of the general cases, part I

We split the goal of this chapter, p.4 Theorem 1.4, into two propositions:

**Proposition 4.4 (General cases, part I)**
*Let $A$ be an algebraic conjugate of $q$. Suppose one of the following holds:*

- $b_i/b_m \in \mathbb{Q}$ *for all* $i$;
- $m \le 2$;
- $m = 3$ *and* $\mathbb{L}_\ell(A, \mathfrak{w})$ *does not have code* $\langle 1302\rangle$ *or* $\langle 1032\rangle$ *for all nonzero* $\mathfrak{w}$ *and all* $\ell \ge 1$.

*If $\Phi$ satisfies the FTC, then $|A| < 1$.*

**Proposition 4.5 (General cases, part II)**
*Let $A$ be an algebraic conjugate of $q$. Suppose $m = 3$. If $\Phi$ satisfies the FTC, then $|A| < 1$.*

We prove the former in this section and handle the latter in the next. The following proposition, which we call "basic move", will be used repeatedly in our argument.

**Proposition 4.6 (Basic move)**
*Let $A$ be an algebraic conjugate of $q$ and $G = G(A) \subseteq \mathbb{C}$ a set of complex numbers. Suppose there exist $y \in Y \cap (0,1)$ and $\varpi \in \mathbb{C}\setminus\{0\}$ such that*

$$\Re\left(\sigma_A(y)\varpi\right) < 0 \quad \text{and} \quad -\Re\left(\sigma_A(y)\varpi\right) + \Re\left(\zeta\varpi\right) > 0 \ \text{ for all } \zeta \in G.$$

*Let $\mathfrak{w} := \varpi$. Suppose further that given any $k_0 \ge 1$, there exists $\mathcal{S} = \mathcal{S}(\mathfrak{w}, k_0) \subseteq (\mathsf{D}(q) - \mathsf{D}(q))$ satisfying the following properties:*

- $q - 1 \in \mathcal{S}$ *and* $\min \mathcal{S} \le 0$;
- $\mathrm{gap}_{\max}(\mathcal{S}) \le 1$;
- *given any $\delta \in \mathcal{S}$, there exists $\zeta \in G$ such that*

$$(♗) \qquad \sum_{i=1}^{k_0-1} \Re\left(\frac{(A-1)\mathfrak{w}}{A^i}\right) + \Re\left(\frac{\sigma_A(\delta)\mathfrak{w}}{A^{k_0}}\right) \ge \Re\left(\zeta\mathfrak{w}\right).$$

*If $\Phi$ satisfies the FTC, then $|A| < 1$.*

**Proof**
It is proved by contradiction. Suppose on the contrary $|A| \ge 1$. By hypothesis, we have

$$\Re\left(\sigma_A(y)\mathfrak{w}\right) < 0 \quad \text{and} \quad -\Re\left(\sigma_A(y)\mathfrak{w}\right) + \Re\left(\zeta\mathfrak{w}\right) > 0 \ \text{ for all } \zeta \in G. \tag{4.7}$$

With $\mathbb{L}_i = \mathbb{L}_i(A, \mathfrak{w})$ and $\mathbb{H}_i^+ = \mathbb{H}_i^+(A, \mathfrak{w})$, let $k_0$ be the smallest non-negative integer such that

$$y \leq \sum_{i=1}^{k_0} \frac{q-1}{q^i} + \sum_{i=k_0+1}^{\infty} \frac{\max \mathbb{H}_i^+}{q^i}.$$

Let $k_0^* := \begin{cases} 1 & \text{if } k_0 = 0 \\ k_0 & \text{otherwise} \end{cases}$. For each $i \geq k_0^*$, we define a set $\mathbb{E}_i \subseteq (\mathsf{D}(q) - \mathsf{D}(q))$. If $k_0 = 0$, then we define $\mathbb{E}_i := \mathbb{H}_i^+$ for all $i \geq k_0^*$. Else if $k_0 \geq 1$, then we take

$$\mathbb{E}_i := \begin{cases} \mathcal{S}(\mathfrak{w}, k_0) & \text{if } i = k_0^* \\ \mathbb{H}_i^+ & \text{if } i > k_0^*. \end{cases}$$

Since $q-1$ belongs to $\mathcal{S}$, we have

$$y - \sum_{i=1}^{k_0^*-1} \frac{q-1}{q^i} \leq \sum_{i=k_0^*}^{\infty} \frac{\max \mathbb{E}_i}{q^i}.$$

As $\min \mathcal{S} \leq 0$ and $0 \in \mathbb{H}_i^+$, by $y > 0$ and the definition of $k_0$ we also have

$$\begin{aligned} y - \sum_{i=1}^{k_0^*-1} \frac{q-1}{q^i} &> \begin{cases} 0 & \text{if } k_0 = 0 \\ \left( \sum_{i=1}^{k_0^*-1} \frac{q-1}{q^i} + \sum_{i=k_0^*}^{\infty} \frac{\max \mathbb{H}_i^+}{q^i} \right) - \sum_{i=1}^{k_0^*-1} \frac{q-1}{q^i} & \text{if } k_0 \geq 1 \end{cases} \\ &\geq \sum_{i=k_0^*}^{\infty} \frac{\min \mathbb{E}_i}{q^i}. \end{aligned}$$

Moreover, in view of the structure of $\mathbb{H}_i^+$ (p.13 Lemma 2.4) and the hypothesis on $\mathcal{S}$, we have

$$\operatorname{gap}_{\max}(\mathbb{E}_\ell) \leq 1 = q^\ell \sum_{i=\ell+1}^{\infty} \frac{\max \mathbb{E}_i - \min \mathbb{E}_i}{q^i}$$

for all $\ell \geq k_0^*$. Thus, our lazy algorithm (p.12 Proposition 2.3), applied to $x := y - \sum_1^{k_0-1} \frac{q-1}{q^i}$, gives

$$y - \sum_{i=1}^{k_0-1} \frac{q-1}{q^i} = \sum_{i=k_0^*}^{\infty} \frac{t_i}{q^i} \tag{4.8}$$

with $t_i \in \mathbb{E}_i$ for all $i \geq k_0^*$.

Define a real number $c_+ \in \mathbb{R}$ by

$$c_+ := \begin{cases} -\Re\left(\sigma_A(y)\mathfrak{w}\right) & \text{if } k_0 = 0 \\ -\Re\left(\sigma_A(y)\mathfrak{w}\right) + \sum_1^{k_0-1} \Re\left(\frac{(A-1)\mathfrak{w}}{A^i}\right) + \Re\left(\frac{\sigma_A(t_{k_0})\mathfrak{w}}{A^{k_0}}\right) & \text{otherwise.} \end{cases} \tag{4.9}$$

We claim that $c_+$ is positive from this definition. Note that by (4.7), $c_+$ is positive if $k_0 = 0$. Else if $k_0 \geq 1$, then as $t_{k_0} \in \mathcal{S}(\mathfrak{w}, k_0)$, so by the hypothesis on $\mathcal{S}$, there exists $\zeta \in G$ such that $c_+ \geq -\Re\left(\sigma_A(y)\mathfrak{w}\right) + \Re\left(\zeta\mathfrak{w}\right)$, which is positive by (4.7). This justifies the claim.

We proceed to handle the cases $|A| > 1$ and $|A| = 1$ separately.

Case $|A| > 1$: In this case, as a consequence of the FTC (p.11 Proposition 2.2), we obtain from (4.8) and (4.9) that

$$-c_+ = \Re\left(\sum_{i=k_0+1}^{\infty} \frac{\sigma_A(t_i)}{A^i}\mathfrak{w}\right).$$

As $t_i \in \mathbb{H}_i^+ \subseteq \mathbb{L}_i$ for all $i \geq k_0 + 1$, the R.H.S. is $\geq 0$, which contradicts $c_+ > 0$.

Case $|A| = 1$: In this case, we can apply p.35 Lemma 4.3 with $K_0 := k_0$ to reach a contradiction.

**Q.E.D.**

To satisfy the conditions of the basic move, we have the following result.

**Lemma 4.7 (Availability of $\mathcal{S}$ for simple patterns)**

*Let $A$ be an algebraic conjugate of $q$, $\mathfrak{w} \in \mathbb{C} \setminus \{0\}$ a nonzero complex number, and $\ell$ a positive integer. Suppose one of the following holds:*

- $b_i/b_m \in \mathbb{Q}$ *for all* $i$*;*
- $m \leq 2$*;*
- $m = 3$ *and* $\mathbb{L}_\ell = \mathbb{L}_\ell(A, \mathfrak{w})$ *does not have code* $\langle 1302\rangle$ *or* $\langle 1032\rangle$*.*

*Then there exists* $\mathbb{S}_\ell = \mathbb{S}_\ell(A, \mathfrak{w}) \subseteq (\mathsf{D}(q) - \mathsf{D}(q))$ *such that*

1. $q - 1 \in \mathbb{S}_\ell$ *and* $\min \mathbb{S}_\ell \leq 0$*;*
2. $\mathrm{gap}_{\max}(\mathbb{S}_\ell) \leq 1$*;*
3. *for all* $\delta \in \mathbb{S}_\ell$*, at least one of the following holds:*

   (i) $\Re\left(\frac{\sigma_A(\delta)\mathfrak{w}}{A^\ell}\right) \geq 0$.  (ii) $\Re\left(\frac{\sigma_A(\delta)\mathfrak{w}}{A^\ell}\right) \geq \Re\left(\frac{(A-1)\mathfrak{w}}{A^\ell}\right)$.

**Proof**

If $b_i/b_m \in \mathbb{Q}$ for all $i$, then $\mathbb{L}_\ell$ has code $\langle m\,(m-1)\cdots 1\,0\rangle$ or $\langle 0\,1\cdots (m-1)\,m\rangle$. Hence, for $\mathbb{S}_\ell := \{[m; *]\}$, either 3(i) or 3(ii) holds for all $\delta \in \mathbb{S}_\ell$. The case $m = 1$ is similar. If $m = 2$, we can use

$$\mathbb{S}_\ell := \begin{cases} \{[2;0]\,, [2;1]\,, 0\} & \text{if } \mathbb{L}_\ell \text{ has code } \langle 210\rangle\,, \langle 201\rangle\,, \langle 021\rangle \text{ or } \langle 012\rangle \\ \{[2;0]\,, [1;0]\,, 0\} & \text{if } \mathbb{L}_\ell \text{ has code } \langle 102\rangle \text{ or } \langle 120\rangle. \end{cases}$$

Lastly, for $m = 3$, the possible codes of $\mathbb{L}_\ell$ are

$$\begin{array}{cccccc} \langle 0123\rangle & \langle 0132\rangle & \langle 0213\rangle & \langle 0231\rangle & \langle 0312\rangle & \langle 0321\rangle \\ \langle 1023\rangle & \cancel{\langle 1032\rangle} & \langle 1203\rangle & \langle 1230\rangle & \cancel{\langle 1302\rangle} & \langle 1320\rangle \\ \langle 2013\rangle & \langle 2031\rangle & \langle 2103\rangle & \langle 2130\rangle & \langle 2301\rangle & \langle 2310\rangle \\ \langle 3012\rangle & \langle 3021\rangle & \langle 3102\rangle & \langle 3120\rangle & \langle 3201\rangle & \langle 3210\rangle\,. \end{array}$$

We can define $\mathbb{S}_\ell$ according to the following table:

| Code of $\mathbb{L}_\ell$ | $\mathbb{S}_\ell$ |
|---|---|
| $\langle 0\cdots\rangle$ | $\{[3;0],[3;1],[3;2],0\}$ |
| $\langle 1023\rangle,\langle 1203\rangle,\langle 1230\rangle$ or $\langle 1320\rangle$ | $\{[3;0],[2;0],[1;0],0\}$ |
| $\langle 2\cdots\rangle$ | $\{[3;0],[2;0],[2;1],0\}$ |
| $\langle 3\cdots\rangle$ | $\{[3;0],[3;1],[3;2],0\}$ |

**Q.E.D.**

We are now ready to achieve the target of this section.

**Proof of the general cases, part I (p.36 Proposition 4.4)**

It is proved by contradiction. Suppose on the contrary $|A| \geq 1$. Let $y \in Y \cap (0,1)$ and $\varpi_\varepsilon \in \mathbb{C} \setminus \{0\}$ be obtained from p.17 Proposition 2.8(d), where $\varepsilon := 0.5$ (the value of $\varepsilon$ is not important in this proof). Let $G \subseteq \mathbb{C}$ be defined by

$$G := \left\{1 - \frac{1}{A^\ell} : \ell \geq 0\right\}.$$

By the assumption $|A| \geq 1$, we have $G \subseteq D_1(1)$. Therefore, by the properties of $y$ and $\varpi_\varepsilon$, we have

$$\Re\left(\sigma_A(y)\varpi_\varepsilon\right) < 0 \quad \text{and} \quad -\Re\left(\sigma_A(y)\varpi_\varepsilon\right) + \Re\left(\zeta\varpi_\varepsilon\right) > 0 \ \text{ for all } \zeta \in G.$$

Let $\mathfrak{w} := \varpi_\varepsilon$. Given $k_0 \geq 1$, we take $\mathcal{S} = \mathcal{S}(\mathfrak{w}, k_0) := \mathbb{S}_{k_0}(A, \mathfrak{w})$ using the preceding Lemma 4.7. We apply the basic move (Proposition 4.6) and it remains to check (♙).

This can be shown as follows. Given $\delta \in \mathcal{S}$, by the property of $\mathbb{S}_{k_0}$, one of the following holds:

$$\text{(i) } \Re\left(\frac{\sigma_A(\delta)}{A^{k_0}}\mathfrak{w}\right) \geq 0; \quad \text{or} \quad \text{(ii) } \Re\left(\frac{\sigma_A(\delta)}{A^{k_0}}\mathfrak{w}\right) \geq \Re\left(\frac{(A-1)}{A^{k_0}}\mathfrak{w}\right).$$

Hence, the L.H.S. of (♙) is $\geq \Re\left(\left(1 - \frac{1}{A^j}\right)\mathfrak{w}\right)$, where $j$ equals $k_0 - 1$ or $k_0$. This fits the definition of $G$ and (♙).

**Q.E.D.**

## 4.3 Proof of the general cases, part II

To accomplish the goal of this chapter, it remains to prove part II of the general cases (p.36 Proposition 4.5). The focus of this section therefore is the case $m = 3$ (i.e. $\Phi = \{\rho x + \partial_i \rho\}_{i=0}^3$ is a 4-tuple). Let $a, b \in \mathbb{Q}[q]$ be such that

$$\begin{array}{c|c|c|c} \partial_0 & \partial_1 & \partial_2 & \partial_3 \\ \hline 0 & a & q-1-b & q-1 \end{array}.$$

Define two subsets $\mathcal{S}_a$, $\mathcal{S}_b$ of $(\mathsf{D}(q) - \mathsf{D}(q))$ by

$$\mathcal{S}_a \stackrel{\text{def}}{=} \{[3;0],[3;1],[3;2],0\} = \{q-1,\ q-1-a,\ b,\ 0\},$$
$$\mathcal{S}_b \stackrel{\text{def}}{=} \{[3;0],[2;0],[1;0],0\} = \{q-1,\ q-1-b,\ a,\ 0\}.$$

Also, given an algebraic conjugate $A$ of $q$, define two sets $G_a, G_b$ by

$$G_a = G_a(A) \stackrel{\text{def}}{=\!=} \left\{1 - \frac{1}{A^\ell} : \ell \geq 0\right\} \cup \left\{1 - \frac{1+\sigma_A(a)}{A^\ell} : \ell \geq 1\right\},$$
$$G_b = G_b(A) \stackrel{\text{def}}{=\!=} \left\{1 - \frac{1}{A^\ell} : \ell \geq 0\right\} \cup \left\{1 - \frac{1+\sigma_A(b)}{A^\ell} : \ell \geq 1\right\}.$$

Lemma 4.7 (p.38) reveals that there are only two code representations that are resistant to our previous strategy. They are $\langle 1302\rangle$ and $\langle 1032\rangle$. We highlight the following heuristic: when using the basic move (p.36 Proposition 4.6) for the case $m = 3$, in many situations we are able to conclude that $\mathbb{L}_{k_0}(A, \mathfrak{w})$ has code $\langle 1302\rangle$ or $\langle 1032\rangle$, and that $\delta = q - 1 - a$ or $q - 1 - b$ in (♗) according to $\mathcal{S} = \mathcal{S}_a$ or $\mathcal{S}_b$.

**Lemma 4.8 (4-tuple: Upper bounds of $|A|$)**

*Let $\Phi$ be a 4-tuple and $A$ an algebraic conjugate of $q$. Suppose $\Phi$ satisfies the FTC. If $|A| \geq 1$, then*

(i) $|A| < |1 + \sigma_A(a)|$; (ii) $|A| < |1 + \sigma_A(b)|$.

**Proof**

It is proved by contradiction. To begin with, let $y \in Y \cap (0,1)$ and $\varpi_\varepsilon \in \mathbb{C} \setminus \{0\}$ be obtained from p.17 Proposition 2.8(d), where $\varepsilon := 0.5$ (the value of $\varepsilon$ is not important in this proof).

(i) Suppose on the contrary $|A| \geq |1 + \sigma_A(a)|$. Let $G := G_a(A)$. By our assumption and by $|A| \geq 1$, we have $G \subseteq D_1(1)$. Therefore, by the properties of $y$ and $\varpi_\varepsilon$, we have

$$\text{(4.10)}\quad \Re\left(\sigma_A(y)\varpi_\varepsilon\right) < 0 \quad\text{and}\quad -\Re\left(\sigma_A(y)\varpi_\varepsilon\right) + \Re\left(\zeta\varpi_\varepsilon\right) > 0 \text{ for all } \zeta \in G.$$

We use the basic move (p.36 Proposition 4.6) to reach a contradiction. Let $\mathfrak{w} := \varpi_\varepsilon$. Given $k_0 \geq 1$, take

$$\mathcal{S}(\mathfrak{w}, k_0) := \begin{cases} \mathbb{S}_{k_0}(A, \mathfrak{w}) & \text{if } \mathbb{L}_{k_0}(A, \mathfrak{w}) \text{ does not have code } \langle 1302\rangle \text{ or } \langle 1032\rangle \\ \mathcal{S}_a & \text{otherwise,} \end{cases}$$

where $\mathbb{S}_{k_0}(A, \mathfrak{w})$ is from p.38 Lemma 4.7. It remains to check (♗).

Given $\delta \in \mathcal{S} = \mathcal{S}(\mathfrak{w}, k_0)$, if one of the following holds:

$$\text{(4.11)}\qquad \text{(i) } \Re\left(\frac{\sigma_A(\delta)\mathfrak{w}}{A^{k_0}}\right) \geq 0; \quad\text{or}\quad \text{(ii) } \Re\left(\frac{\sigma_A(\delta)\mathfrak{w}}{A^{k_0}}\right) \geq \Re\left(\frac{(A-1)\mathfrak{w}}{A^{k_0}}\right),$$

then the L.H.S. of (♗) is $\geq \Re\left(\left(1 - \dfrac{1}{A^j}\right)\mathfrak{w}\right)$, where $j$ equals $k_0 - 1$ or $k_0$, whence (♗) is fulfilled by the definition of $G$. Else, suppose (4.11) does not hold. In view of the properties of $\mathbb{S}_{k_0}(A, \mathfrak{w})$ from Lemma 4.7, we see that $\mathbb{L}_{k_0}(A, \mathfrak{w})$ has code $\langle 1302\rangle$ or $\langle 1032\rangle$, and $\delta \in \mathcal{S}_a = \{[3;0], [3;1], [3;2], 0\}$. As $\delta$ does not satisfy (4.11), the code of $\mathbb{L}_{k_0}(A, \mathfrak{w})$ forces $\delta = [3;1] = q-1-a$. Here $[3;1]$ may be called a "compulsory digit". It follows that the L.H.S. of (♙) equals

$$\Re\left(\left(1-\frac{1+\sigma_A(a)}{A^{k_0}}\right)\mathfrak{w}\right).$$

Again (♙) is fulfilled by the definition of $G$. This completes the proof of part(i).

(ii) To prove this, we just need to repeat the proof of part(i) with $b$ in place of $a$, so that we use $G_b, \mathcal{S}_b$ instead of $G_a, \mathcal{S}_a$. The "compulsory digit" will be $[2;0] = q-1-b$ instead of $q-1-a$.

**Q.E.D.**

**Lemma 4.9 (4-tuple: Inequalities for $Y_{\text{core}}$)**

*Let $\Phi$ be a 4-tuple and $A$ an algebraic conjugate of $q$. Suppose $\Phi$ satisfies the FTC. If $|A| \geq 1$, then for all $y \in Y_{\text{core}}$, we have*

(i) $\displaystyle\left|\frac{\sigma_A(a)+1}{A}\right| \geq |\sigma_A(y)-1|$;  (ii) $\displaystyle\left|\frac{\sigma_A(b)+1}{A}\right| \geq |\sigma_A(y)-1|$.

**Proof**

It is proved by contradiction. Plainly the statements are true when $y = 1$, so we may only consider $y \in Y \cap (0,1)$.

(i) Suppose on the contrary $\displaystyle\left|\frac{\sigma_A(a)+1}{A}\right| < |\sigma_A(y)-1|$ for some $y \in Y \cap (0,1)$. Noting that $|A| < |1+\sigma_A(a)|$ by the preceding Lemma 4.8, we have

$$1 < |\sigma_A(y)-1|\,. \tag{4.12}$$

Let $\Omega_0 := e^{i\theta}$ be such that $\Re\left((\sigma_A(y)-1)\Omega_0\right) = -|\sigma_A(y)-1|$ and $G := G_a(A)$. We claim that

$$\text{(I) } \Re\left(\sigma_A(y)\Omega_0\right) < 0 \quad\text{and}\quad \text{(II) } -\Re\left(\sigma_A(y)\Omega_0\right) + \Re\left(\zeta\Omega_0\right) > 0 \ \text{ for all } \zeta \in G.$$

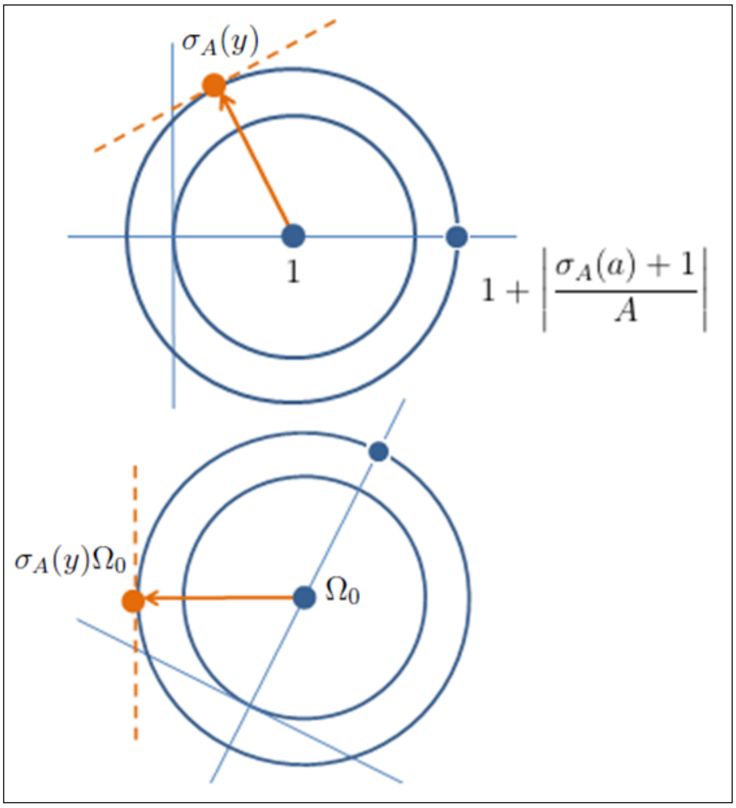
For, if $\zeta = 1 - 1/A^\ell$ with $\ell \geq 0$, then by (4.12)

$$-\Re\left(\sigma_A(y)\Omega_0\right) + \Re\left(\zeta\Omega_0\right) = |\sigma_A(y)-1| - \Re\left(\frac{\Omega_0}{A^\ell}\right) > 1 - \frac{|\Omega_0|}{|A|^\ell} \geq 0.$$

Else if $\zeta = 1 - (1+\sigma_A(a))/A^\ell$ with $\ell \geq 1$, then

$$\begin{aligned}-\Re\left(\sigma_A(y)\Omega_0\right) + \Re\left(\zeta\Omega_0\right) &= |\sigma_A(y)-1| - \Re\left(\frac{1+\sigma_A(a)}{A^\ell}\Omega_0\right)\\ &\geq |\sigma_A(y)-1| - \frac{|1+\sigma_A(a)|}{|A|}.\end{aligned}$$

Noting that the last expression is $> 0$ by our initial assumption, we have verified the claim. It may also be illustrated by the following figure:




Now we are like having (4.10), with $\Omega_0$ in place of $\varpi_\varepsilon$. We can use the same argument as in the proof of Lemma 4.8(i) to reach a contradiction.

(ii) As in the proof of Lemma 4.8(ii), we just need to repeat the argument in part(i) with $b$ in place of $a$.

**Q.E.D.**

**Corollary 4.10 (4-tuple: Asymmetric IFS with $\Re(\sigma_A(a)), \Re(\sigma_A(b)) \geq 0$)**
*Let $\Phi$ be a 4-tuple and $A$ an algebraic conjugate of $q$. Suppose $\Phi$ satisfies the FTC. If $|A| \geq 1$, then*

(i) $a \neq b$;

(ii) $\Re(\sigma_A(a)) \geq 0$ *and* $\Re(\sigma_A(b)) \geq 0$.

**Proof**
It is proved by using the previous two lemmas.

(i) Suppose on the contrary $a = b$. Taking $y := [2; 1] = q-1-2a$ in Lemma 4.9(i), we have

$$|\sigma_A(a)+1| \geq \left|\frac{\sigma_A(a)+1}{A}\right| \geq |A-2-2\sigma_A(a)| \geq 2\,|1+\sigma_A(a)| - |A|\,.$$

On the other hand, we have $|A| < |1+\sigma_A(a)|$ by Lemma 4.8(i). Therefore the above is a contradiction.

(ii) Suppose $\Re(\sigma_A(a)) < 0$. Then in the complex plane, the point $\sigma_A(a)$ is closer to the point $-1$ then 1, whence $|\sigma_A(a)+1| < |\sigma_A(a)-1|$. This contradicts Lemma 4.9(i) nevertheless (with $y := a$). This shows $\Re(\sigma_A(a)) \geq 0$. By a similar argument using $y := [3;2] = b$, we get $\Re(\sigma_A(b)) \geq 0$ too.

**Q.E.D.**

We remark that although part (i) of the preceding corollary sheds some light on the issue, it will not be used directly in our subsequent argument.

**Lemma 4.11 (4-tuple: One way or another)**

*Let $\Phi$ be a 4-tuple and $A$ an algebraic conjugate of $q$. Suppose $\Phi$ satisfies the FTC. If $|A| \geq 1$, and $\Re(A) < 1 + \Re(\sigma_A(a))$, then for $y := [2;1] = q-1-a-b$ the following holds. For each $\ell \geq 1$, there exists $\delta_\ell \in \{a,b\}$ such that*

$$-\Re(\sigma_A(y)) + \Re\left(1 - \frac{1+\sigma_A(\delta_\ell)}{A^\ell}\right) > 0.$$

**Proof**

It is proved by contradiction. Suppose on the contrary there is some $\ell \geq 1$ such that

$$\begin{cases} -\Re(\sigma_A(y)) + \Re\left(1 - \dfrac{1}{A^{\ell-1}}\right) + \Re\left(\dfrac{A-1-\sigma_A(a)}{A^\ell}\right) \leq 0 \\ -\Re(\sigma_A(y)) + \Re\left(1 - \dfrac{1}{A^\ell}\right) - \Re\left(\dfrac{\sigma_A(b)}{A^\ell}\right) \leq 0. \end{cases}$$

Let $\xi := 1 - \frac{1}{A^\ell}$. Noting that both $\Re\left(1 - \frac{1}{A^{\ell-1}}\right), \Re\left(1 - \frac{1}{A^\ell}\right) \geq 0$, we have

(4.13)
$$\begin{cases} -\Re([A-1-\sigma_A(a)]\xi) + \Re(\sigma_A(b)) = -\Re(\sigma_A(y)) + \Re\left(\dfrac{A-1-\sigma_A(a)}{A^\ell}\right) \leq 0 \\ -\Re(A-1-\sigma_A(a)) + \Re(\sigma_A(b)\xi) = -\Re(\sigma_A(y)) - \Re\left(\dfrac{\sigma_A(b)}{A^\ell}\right) \leq 0. \end{cases}$$

As $\Re(A-1-\sigma_A(a)) < 0$ by hypothesis and $\Re(\xi) \geq 0$, the first inequality gives

$$\begin{aligned} \Re(\sigma_A(b)) &\leq \Re([A-1-\sigma_A(a)]\xi) \\ &= \Re(A-1-\sigma_A(a))\,\Re(\xi) - \Im(A-1-\sigma_A(a))\,\Im(\xi) \\ &\leq -\Im(A-1-\sigma_A(a))\,\Im(\xi). \end{aligned}$$

Noting that $\Re(\sigma_A(b)) \geq 0$ by Corollary 4.10, we see that $\Im(A-1-\sigma_A(a))$ and $\Im(\xi)$ have different $\pm$ signs (including zero). Similarly, the second inequality of (4.13) gives

$$\begin{aligned} \Re(A-1-\sigma_A(a)) &\geq \Re(\sigma_A(b)\xi) \\ &= \Re(\sigma_A(b))\,\Re(\xi) - \Im(\sigma_A(b))\,\Im(\xi) \\ &\geq -\Im(\sigma_A(b))\,\Im(\xi). \end{aligned}$$

As $\Re(A-1-\sigma_A(a)) < 0$ by hypothesis, we see that $\Im(\sigma_A(b))$ and $\Im(\xi)$ have the same $\pm$ sign. We conclude that $\Im(A-1-\sigma_A(a))$ and $\Im(\sigma_A(b)) = \Im(1+\sigma_A(b))$

have different $\pm$ signs (including zero). Together with $\Re(A-1-\sigma_A(a)) < 0 \leq \Re(\sigma_A(b)) < \Re(1+\sigma_A(b))$, we have

$$\begin{cases} |\Im(A-1-\sigma_A(a)-\sigma_A(b)-1)| \geq |\Im(1+\sigma_A(b))| \\ |\Re(A-1-\sigma_A(a)-\sigma_A(b)-1)| > |\Re(1+\sigma_A(b))|, \end{cases}$$

whence

$$|A-1-\sigma_A(a)-\sigma_A(b)-1| > |1+\sigma_A(b)|.$$

However, one of the inequalities for $Y_{\mathrm{core}}$ (Lemma 4.9) gives

$$|1+\sigma_A(b)| \geq \frac{|1+\sigma_A(b)|}{|A|} \geq |\sigma_A(y)-1|$$

(as $y = q-1-a-b = [2;1]$ belongs to $Y_{\mathrm{core}}$). This shows a contradiction and completes the proof.

**Q.E.D.**

**<u>Lemma 4.12 (4-tuple: Dominance of $\Re(A)$)</u>**

*Let $\Phi$ be a 4-tuple and $A$ an algebraic conjugate of $q$. Suppose $\Phi$ satisfies the FTC. If $|A| \geq 1$, then*

(i) $\Re(A) \geq 1 + \Re(\sigma_A(a))$; (ii) $\Re(A) \geq 1 + \Re(\sigma_A(b))$.

**<u>Proof</u>**

We prove (i) first and it is proved by contradiction. Suppose on the contrary $\Re(A) < 1 + \Re(\sigma_A(a))$. Let $y := [2;1] = q-1-a-b$. As $\Re(\sigma_A(b)) \geq 0$ by Corollary 4.10, we have $\Re(\sigma_A(y)) = \Re(A-1-\sigma_A(a)) - \Re(\sigma_A(b)) < 0$, so $y \in Y \cap (0,1)$. Define $G \subseteq \mathbb{C}$ by

$$G := \left\{1 - \frac{1}{A^\ell} : \ell \geq 0\right\} \cup \left\{1 - \frac{1+\sigma_A(\delta_\ell)}{A^\ell} : \ell \geq 1\right\},$$

where $\delta_\ell$ is given by the preceding Lemma 4.11. By that result, we have

$$\Re(\sigma_A(y)) < 0 \quad \text{and} \quad -\Re(\sigma_A(y)) + \Re(\zeta) > 0 \;\text{ for all } \zeta \in G.$$

Once again we use the basic move (p.36 Proposition 4.6) to reach a contradiction. Let $\mathfrak{w} := 1$. Given $k_0 \geq 1$, take

$$\mathcal{S}(\mathfrak{w}, k_0) := \begin{cases} \mathbb{S}_{k_0}(A, \mathfrak{w}) & \text{if } \mathbb{L}_{k_0}(A, \mathfrak{w}) \text{ does not have code } \langle 1302 \rangle \text{ or } \langle 1032 \rangle \\ \mathcal{S}_{\delta_{k_0}} & \text{otherwise,} \end{cases}$$

where $\mathbb{S}_{k_0}(A, \mathfrak{w})$ is from p.38 Lemma 4.7. It remains to check (♙).

Given $\delta \in \mathcal{S}$, if one of the following holds:

$$\text{(I)} \;\; \Re\left(\frac{\sigma_A(\delta)\mathfrak{w}}{A^{k_0}}\right) \geq 0; \quad \text{or} \quad \text{(II)} \;\; \Re\left(\frac{\sigma_A(\delta)\mathfrak{w}}{A^{k_0}}\right) \geq \Re\left(\frac{(A-1)\mathfrak{w}}{A^{k_0}}\right), \tag{4.14}$$

then the L.H.S. of (♙) is $\geq \Re\left(\left(1 - \frac{1}{A^j}\right)\mathfrak{w}\right)$, where $j$ equals $k_0 - 1$ or $k_0$, whence (♙) is fulfilled by the definition of $G$. Else, suppose (4.14) does not hold. In view of the properties of $\mathbb{S}_{k_0}(A, \mathfrak{w})$ from Lemma 4.7, we see that $\mathbb{L}_{k_0}(A, \mathfrak{w})$ has code $\langle 1302 \rangle$ or $\langle 1032 \rangle$, and

$$\delta \in \begin{cases} \mathcal{S}_a = \{[3;0], [3;1], [3;2], 0\} & \text{if } \delta_{k_0} = a \\ \mathcal{S}_b = \{[3;0], [2;0], [1;0], 0\} & \text{if } \delta_{k_0} = b. \end{cases}$$

As $\delta$ does not satisfy (4.14), we have

$$\delta = \begin{cases} [3;1] & \text{if } \delta_{k_0} = a \\ [2;0] & \text{if } \delta_{k_0} = b, \end{cases}$$

so that $\delta = q - 1 - \delta_{k_0}$. It follows that the L.H.S. of (♙) equals

$$\Re\left(\left(1 - \frac{1 + \sigma_A(\delta_{k_0})}{A^{k_0}}\right) \mathfrak{w}\right).$$

Again (♙) is fulfilled by the definition of $G$. This completes the proof of part(i).

To get a proof of part (ii), we just need to interchange the roles of $a$ and $b$ in the previous argument.

**Q.E.D.**

**Corollary 4.13 (4-tuple: Requirement of configuration)**
*Let $\Phi$ be a 4-tuple and $A$ an algebraic conjugate of $q$. Suppose $\Phi$ satisfies the FTC. If $|A| \geq 1$, then*

(i) $\Re(A) \geq 1$;

(ii) $|A| \neq 1$;

(iii) $|\Im(A)| < |\Im(\sigma_A(a))|$ *and* $|\Im(A)| < |\Im(\sigma_A(b))|$;

(iv) $\Re(\sigma_A(a)) > 0$ *and* $\Re(\sigma_A(b)) > 0$.

**Proof**

We have $\Re(\sigma_A(a)) \geq 0$ by Corollary 4.10, hence the preceding Lemma 4.12 gives (i). (ii) follows from (i) as $A \neq 1$. For (iii), as we have the upper bound $|A| < |1 + \sigma_A(a)|$ (Lemma 4.8), and Lemma 4.12 shows the dominance of $\Re(A)$, therefore $|\Im(A)| < |\Im(1 + \sigma_A(a))| = |\Im(\sigma_A(a))|$. Similarly we have $|\Im(A)| < |\Im(\sigma_A(b))|$. For (iv), taking $y := [1;0] = a$ in one of the inequalities for $Y_{\text{core}}$ (Lemma 4.9), we have

$$|\sigma_A(a) - 1| \leq \frac{|\sigma_A(a) + 1|}{|A|}.$$

Since $|A| > 1$ by (ii), we have $|\sigma_A(a) - 1| < |\sigma_A(a) + 1|$, whence $\Re(\sigma_A(a)) > 0$. The inequality for $b$ can be proved similarly using $y := [3;2] = b$.

**Q.E.D.**

By part (ii) of the preceding result, from now on we can restrict our attention to the case $|A| > 1$.

**Lemma 4.14 (4-tuple: Arguments)**

*Let $\Phi$ be a 4-tuple and $A$ an algebraic conjugate of $q$. Suppose $\Phi$ satisfies the FTC. Write*

$$\begin{aligned} A &=: r_A \exp(i\theta_A), \\ 1 + \sigma_A(a) &=: r_\alpha \exp(i\theta_\alpha), \\ 1 + \sigma_A(b) &=: r_\beta \exp(i\theta_\beta), \end{aligned}$$

*where $\theta_A, \theta_\alpha, \theta_\beta \in [-\pi/2, \pi/2]$ by Corollary 4.13. If $|A| > 1$ and $|\theta_\alpha - \theta_\beta| \leq \pi/2$, then we have $\theta_\alpha \neq \theta_\beta$, and $\theta_A$ is strictly between $\theta_\alpha$ and $\theta_\beta$. That is, $\theta_A \in (\min\{\theta_\alpha, \theta_\beta\}, \max\{\theta_\alpha, \theta_\beta\})$.*

**Proof**

Using $y := [2; 1]$ in the inequalities for $Y_{\text{core}}$ (Lemma 4.9), we have

$$|A - 1 - \sigma_A(a) - 1 - \sigma_A(b)|^2 < \min\left\{|1 + \sigma_A(a)|^2, |1 + \sigma_A(b)|^2\right\}. \tag{4.15}$$

The L.H.S. is

$$\begin{aligned} &= (r_A \cos\theta_A - r_\alpha \cos\theta_\alpha - r_\beta \cos\theta_\beta)^2 + (r_A \sin\theta_A - r_\alpha \sin\theta_\alpha - r_\beta \sin\theta_\beta)^2 \\ &= r_A^2 + r_\alpha^2 + r_\beta^2 + 2r_\alpha r_\beta(\cos\theta_\alpha \cos\theta_\beta + \sin\theta_\alpha \sin\theta_\beta) \\ &\qquad - 2r_\alpha r_A(\cos\theta_\alpha \cos\theta_A + \sin\theta_\alpha \sin\theta_A) - 2r_\beta r_A(\cos\theta_\beta \cos\theta_A + \sin\theta_\beta \sin\theta_A) \\ &= r_A^2 + r_\alpha^2 + r_\beta^2 + 2r_\alpha r_\beta \cos(\theta_\alpha - \theta_\beta) - 2r_\alpha r_A \cos(\theta_\alpha - \theta_A) - 2r_\beta r_A \cos(\theta_\beta - \theta_A). \end{aligned}$$

By hypothesis, we have $\cos(\theta_\alpha - \theta_\beta) \geq 0$. Hence, if $\cos(\theta_\alpha - \theta_\beta) \geq \cos(\theta_\alpha - \theta_A)$, then by $r_\beta > r_A$ (Lemma 4.8) we have

$$2r_\alpha r_\beta \cos(\theta_\alpha - \theta_\beta) - 2r_\alpha r_A \cos(\theta_\alpha - \theta_A) \geq 0,$$

whence the L.H.S. of (4.15) is

$$\begin{aligned} &\geq r_A^2 + r_\alpha^2 + r_\beta^2 - 2r_\beta r_A \cos(\theta_\beta - \theta_A) \\ &\geq r_A^2 + r_\alpha^2 + r_\beta^2 - 2r_\beta r_A \geq r_\alpha^2 = |1 + \sigma_A(a)|^2. \end{aligned}$$

This contradicts (4.15). We get a similar contradiction if $\cos(\theta_\alpha - \theta_\beta) \geq \cos(\theta_\beta - \theta_A)$. Therefore, we see that

$$\begin{cases} \cos(\theta_\alpha - \theta_\beta) < \cos(\theta_\alpha - \theta_A) \\ \cos(\theta_\alpha - \theta_\beta) < \cos(\theta_\beta - \theta_A). \end{cases}$$

As $\theta_A, \theta_\alpha, \theta_\beta \in [-\pi/2, \pi/2]$, using a picture we conclude that $\theta_A$ is between $\theta_\alpha$ and $\theta_\beta$. As the above inequalities are strict, we have $\theta_\alpha \neq \theta_\beta$ and therefore $\theta_A \in (\min\{\theta_\alpha, \theta_\beta\}, \max\{\theta_\alpha, \theta_\beta\})$.

**Q.E.D.**

**Corollary 4.15 (4-tuple: $\sigma_A(a)$ and $\sigma_A(b)$ are separated by the real axis)**
*Let $\Phi$ be a 4-tuple and $A$ an algebraic conjugate of $q$. Suppose $\Phi$ satisfies the FTC. If $|A| > 1$, then $\Im(\sigma_A(a))$ and $\Im(\sigma_A(b))$ have different $\pm$ signs.*

**Proof**
It is proved by contradiction. Firstly, suppose both $\Im(\sigma_A(a))$ and $\Im(\sigma_A(b))$ are $\geq 0$. By the configuration requirement (Corollary 4.13) and the dominance of $\Re(A)$ (Lemma 4.12), we have

$$\begin{cases} \Im(1+\sigma_A(a)), \Im(1+\sigma_A(b)) > |\Im(A)| \\ 1 \leq \Re(1+\sigma_A(a)), \Re(1+\sigma_A(b)) \leq \Re(A), \end{cases}$$

whence

$$\begin{cases} r_\alpha \sin\theta_\alpha,\ r_\beta \sin\theta_\beta > |r_A \sin\theta_A| \\ 1 \leq r_\alpha \cos\theta_\alpha,\ r_\beta \cos\theta_\beta \leq r_A \cos\theta_A, \end{cases}$$

where we have used the same notation as in the preceding Lemma 4.14. Consequently,

$$\begin{cases} \tan\theta_\alpha = \dfrac{r_\alpha \sin\theta_\alpha}{r_\alpha \cos\theta_\alpha} > \dfrac{r_A \sin\theta_A}{r_A \cos\theta_A} = \tan\theta_A \\ \tan\theta_\beta = \dfrac{r_\beta \sin\theta_\beta}{r_\beta \cos\theta_\beta} > \dfrac{r_A \sin\theta_A}{r_A \cos\theta_A} = \tan\theta_A. \end{cases}$$

However, as $\Im(1+\sigma_A(a))$ and $\Im(1+\sigma_A(b))$ are both $\geq 0$, we have $|\theta_\alpha - \theta_\beta| \leq \pi/2$, so $\theta_A$ should be strictly between $\theta_\alpha$ and $\theta_\beta$ according to Lemma 4.14. This shows a contradiction.

A similar argument can be used to handle the case both $\Im(\sigma_A(a))$ and $\Im(\sigma_A(b))$ are $\leq 0$. This completes the proof.

**Q.E.D.**

It is the time to accomplish the goal of this chapter.

**Proof of the general cases, part II (p.36 Proposition 4.5)**
It is proved by contradiction. Suppose on the contrary $|A| \geq 1$. It implies $|A| > 1$ by the configuration requirement (Corollary 4.13). Since $\Phi$ satisfies the FTC, it follows from part I (p.36 Proposition 4.4) that $\mathbb{L}_\ell(A, \mathfrak{w})$ has code $\langle 1302\rangle$ or $\langle 1032\rangle$ for some $\mathfrak{w}$ and $\ell$. Using the definition of the code representation, we see that there exists $\theta \in [0, 2\pi)$ such that one of the following holds:

$$\begin{array}{lccccccc} (1)\ \Re\left(\sigma_A(\partial_1)e^{i\theta}\right) & > & \Re\left(\sigma_A(\partial_3)e^{i\theta}\right) & \geq & 0 & > & \Re\left(\sigma_A(\partial_2)e^{i\theta}\right), & \text{or} \\ (2)\ \Re\left(\sigma_A(\partial_1)e^{i\theta}\right) & \geq & 0 & > & \Re\left(\sigma_A(\partial_3)e^{i\theta}\right) & \geq & \Re\left(\sigma_A(\partial_2)e^{i\theta}\right). & \end{array}$$

By abuse of notation, let us call the first situation as $\langle 1302\rangle$ and the second as

$\langle 1032 \rangle$. Accordingly, we have four inequalities:

$$
\begin{array}{lll}
(4.16) & & \\
\quad \langle 1 \cdots 3 \cdots \rangle & \Rightarrow & \Re\left(\sigma_A(a)e^{i\theta}\right) > \Re\left(\sigma_A(q-1)e^{i\theta}\right) \\
& \Rightarrow & 0 > \cos\theta \cdot \Re\left(A - 1 - \sigma_A(a)\right) - \sin\theta \cdot \Im\left(A - 1 - \sigma_A(a)\right); \\
(4.17) & & \\
\quad \langle 1 \cdots 0 \cdots \rangle & \Rightarrow & \Re\left(\sigma_A(a)e^{i\theta}\right) \geq 0 \\
& \Rightarrow & \cos\theta \cdot \Re\left(\sigma_A(a)\right) - \sin\theta \cdot \Im\left(\sigma_A(a)\right) \geq 0; \\
(4.18) & & \\
\quad \langle \cdots 3 \cdots 2 \rangle & \Rightarrow & \Re\left(\sigma_A(q-1)e^{i\theta}\right) \geq \Re\left(\sigma_A(q-1-b)e^{i\theta}\right) \\
& \Rightarrow & \cos\theta \cdot \Re\left(\sigma_A(b)\right) - \sin\theta \cdot \Im\left(\sigma_A(b)\right) \geq 0; \\
(4.19) & & \\
\quad \langle \cdots 0 \cdots 2 \rangle & \Rightarrow & 0 > \Re\left(\sigma_A(q-1-b)e^{i\theta}\right) \\
& \Rightarrow & 0 > \cos\theta \cdot \Re\left(A - 1 - \sigma_A(b)\right) - \sin\theta \cdot \Im\left(A - 1 - \sigma_A(b)\right).
\end{array}
$$

Since $\sigma_A(a), \sigma_A(b)$ are separated by the real axis (Corollary 4.15), so by the configuration requirement (Corollary 4.13(iii)), there are two cases:

$$
\begin{cases}
\Im\left(\sigma_A(a)\right) > 0, \quad \Im\left(\sigma_A(b)\right) < 0 \\
\Im\left(A - 1 - \sigma_A(a)\right) < 0 \\
\Im\left(A - 1 - \sigma_A(b)\right) > 0,
\end{cases}
\qquad \text{or} \qquad
\begin{cases}
\Im\left(\sigma_A(a)\right) < 0, \quad \Im\left(\sigma_A(b)\right) > 0 \\
\Im\left(A - 1 - \sigma_A(a)\right) > 0 \\
\Im\left(A - 1 - \sigma_A(b)\right) < 0.
\end{cases}
$$

We consider the first case only as the second can be studied similarly. By $\Re\left(\sigma_A(a)\right)$, $\Re\left(\sigma_A(b)\right) > 0$ (Corollary 4.13(iv)) and the dominance of $\Re\left(A\right)$ (Lemma 4.12),

$$
\begin{array}{lccccc}
(4.16) \text{ gives} & 0 & > & \cos\theta \cdot [\geq 0] & - & \sin\theta \cdot [< 0], \\
(4.17) \text{ gives} & 0 & \geq & \cos\theta \cdot [< 0] & - & \sin\theta \cdot [< 0], \\
(4.18) \text{ gives} & 0 & \geq & \cos\theta \cdot [< 0] & - & \sin\theta \cdot [> 0], \\
(4.19) \text{ gives} & 0 & > & \cos\theta \cdot [\geq 0] & - & \sin\theta \cdot [> 0].
\end{array}
$$

However, contradictions arise from

the 1st when $\theta \in [0,\ \pi/2]$,
the 2nd when $\theta \in (\pi/2,\ \pi]$,
the 3rd when $\theta \in (\pi,\ 3\pi/2]$,
the 4th when $\theta \in (3\pi/2,\ 2\pi)$.

This completes the proof.

**Q.E.D.**

So, we have finally finished the proof of the general cases. We end this chapter by noting that, as a byproduct, we obtain the following result about pattern-avoiding configurations given by planar point sets.

**Proposition 4.16 (Pattern-avoiding configurations)**
*Suppose $(\zeta_1, \zeta_2, g) \in \mathbb{C}^3$ satisfies the following conditions:*

(i) $\Re(\zeta_1) > 1, \;\; \Re(\zeta_2) > 1.$

(ii) $\Re(g - \zeta_1) \geq 0, \;\; \Re(g - \zeta_2) \geq 0.$

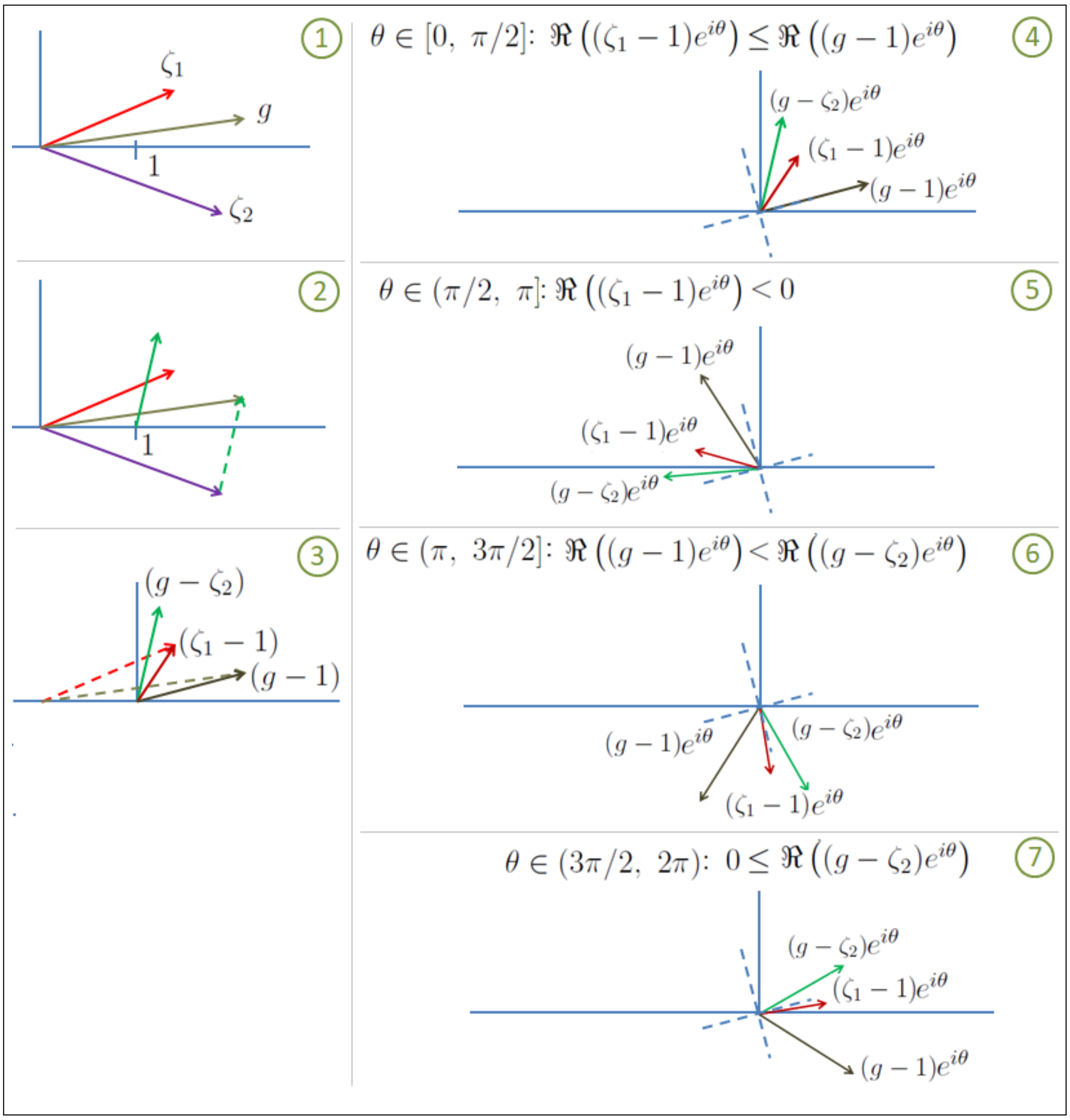
(iii) $|\Im(\zeta_1)| > |\Im(g)|, \;\; |\Im(\zeta_2)| > |\Im(g)|.$

(iv) *$\Im(\zeta_1)$ and $\Im(\zeta_2)$ have different $\pm$ signs.*

*Then for all $\theta \in [0, 2\pi)$, neither (1) nor (2) below holds.*

$$\begin{array}{lllllll} (1)\ \Re\left((\zeta_1-1)e^{i\theta}\right) & > & \Re\left((g-1)e^{i\theta}\right) & \geq & 0 & > & \Re\left((g-\zeta_2)e^{i\theta}\right); \\ (2)\ \Re\left((\zeta_1-1)e^{i\theta}\right) & \geq & 0 & > & \Re\left((g-1)e^{i\theta}\right) & \geq & \Re\left((g-\zeta_2)e^{i\theta}\right). \end{array}$$

The following figure illustrates this result for the case $\Im(\zeta_1) > 0 > \Im(\zeta_2)$.



# Chapter 5

# Examples

In this chapter, we present examples of $\Phi$ satisfying the FTC but the associated $q$ is not a PV number. We use the concept of basic net interval and characteristic vector introduced in [20] in our discussion. The definition can be found in Appendix A (p.81).

As Theorem 1.8 (p.5) states, we demonstrate five examples:

I. Arbitrary $m \geq 5$, $m$ odd, and $A \in \mathbb{R} \cap (1, q)$.

II. Arbitrary $m \geq 5$, $m$ odd, and $A \in \mathbb{R} \cap (-q, -1)$.

III. Arbitrary $m \geq 6$, $m$ even, and $A \in \mathbb{R} \cap (1, q)$.

IV. $m = 4$, $|A| > 1$.

V. $m = 4$, $|A| = 1$.

They are discovered by educated guess and computer experiment. The first three are inspired by the result of Chapter 3, while the last two are by Chapter 4. As this chapter may present overwhelming information, the reader can just take a look at Example I and V for an understanding of the general ideas.

In the final section, we explain in detail how Example IV and V are found. We finish by presenting miscellaneous results which are inspired by the examples.

## 5.1 Example I

Let $m \geq 5$ be an odd integer and $a \in \mathbb{R}$. Consider the situation

$$\begin{cases} \partial_i = i & \text{if } i \in [0,\ (m-3)/2] \\ \partial_i = q - 1 - m + i & \text{if } i \in [(m+3)/2,\ m] \end{cases}$$

and

$$\begin{array}{c|c|c|c||c|c||c|c|c|c} \partial_{(m-3)/2} & \cdots & \partial_1 & \partial_{(m-1)/2} & \partial_m & \partial_0 & \partial_{(m+1)/2} & \partial_{m-1} & \cdots & \partial_{(m+3)/2} \\ \hline (m-3)/2 & \cdots & 1 & a & q-1 & 0 & q-1-a & q-2 & \cdots & q-1-(m-3)/2 \end{array}.$$

We make some hypotheses in light of the proof of Proposition 3.2 (p.27). We assume that $q$ has a real algebraic conjugate $A > 1$, that $\mathbb{P}(A)$ has code

$$\left\langle \frac{m-3}{2}, \cdots, \frac{m-1}{2}, \cdots, m, \cdots, 0, \cdots, \frac{m+1}{2}, \cdots, \frac{m+3}{2} \right\rangle,$$

and that

$$\begin{cases} \sigma_A(\left[\frac{m-1}{2}; \frac{m-3}{2}\right]) = \frac{(A-1)-\frac{m-3}{2}}{A} \\ \sigma_A(\left[\frac{m+1}{2}; \frac{m-1}{2}\right]) = \frac{(A-1)-\frac{m-3}{2}}{A} \\ \sigma_A(\left[\frac{m+3}{2}; \frac{m+1}{2}\right]) = \frac{(A-1)-\frac{m-3}{2}}{A}. \end{cases} \tag{5.1}$$

Adding the equations, we have

$$\begin{aligned} A - m + 2 &= 3 \cdot \frac{A - 1 - \frac{m-3}{2}}{A} \\ \Rightarrow A^2 - mA + 2A &= 3A - 1.5m + 1.5 \\ \Rightarrow A^2 - (m+1)A &+ 1.5m - 1.5 = 0, \end{aligned}$$

whence

$$q, A = \frac{(m+1) \pm \sqrt{(m+1)^2 - 6(m-1)}}{2} = \frac{(m+1) \pm \sqrt{m^2 - 4m + 7}}{2}.$$

As $m \geq 5$, we have $(m+1)^2 \geq (m+1)^2 - 6(m-1) > 0$, so both $q$ and $A$ are positive real numbers. We have $A > 1$ because

$$\begin{aligned} \frac{(m+1) - \sqrt{m^2-4m+7}}{2} > 1 \quad &\Leftarrow \quad (m-1)^2 > (\sqrt{m^2-4m+7})^2 \\ &\Leftarrow \quad 2m - 6 > 0. \end{aligned}$$

A fortiori $q > A > 1$. Note that $q \notin \mathbb{N}$, for otherwise there exists $N \in \mathbb{N}$ such that $N^2 = m^2 - 4m + 7 = (m-2)^2 + 3$, whence

$$\begin{cases} N - (m-2) = 1 \\ N + (m-2) = 3. \end{cases}$$

This gives $N = 2$ and $m = 3$, which is a contradiction. From the quadratic equation, we have $q(q - (m+1)) = 1.5 - 1.5m$, whence $q^{-1} = \dfrac{q-(m+1)}{1.5 - 1.5m}$. Therefore, (5.1) gives

$$a = \frac{m-3}{2} + \frac{q - 1 - \frac{m-3}{2}}{q} = \frac{m-1}{2} + \frac{q-(m+1)}{3} = \frac{2q + m - 5}{6},$$

and so

$$\partial_{(m+1)/2} = q - 1 - a = \frac{4q - m - 1}{6}.$$

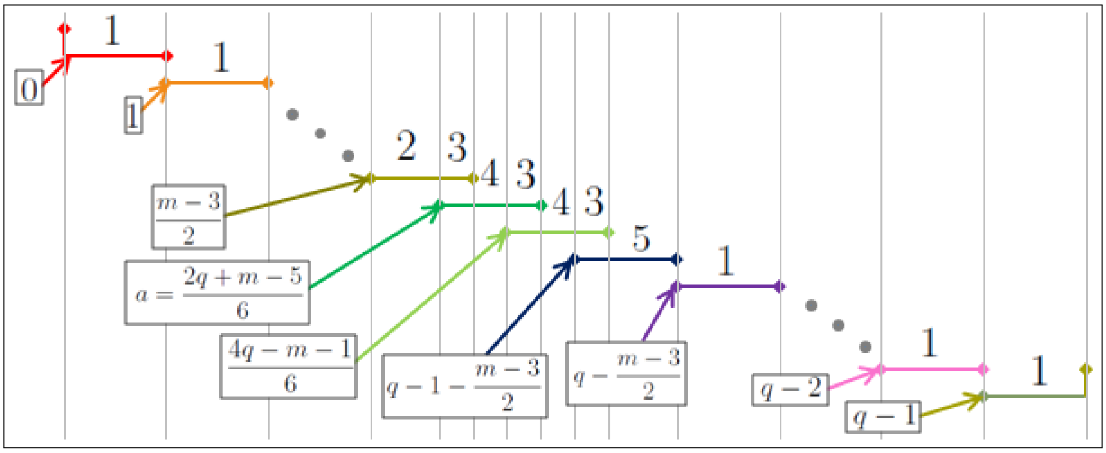



Figure 5.1.1: 1st iteration of Example I.

Since

$$(q-1)-\frac{m-3}{2}=1+\frac{\sqrt{m^2-4m+7}}{2}>0,$$

we see from (5.1) that $\Phi = \{\rho x + \partial_i \rho\}_{i=0}^{m}$ is of the required form of Chapter 2 section 2.1.

We proceed to verify that $\Phi$ satisfies the FTC. Figure 5.1.1 illustrates the 1st iteration. In the figure, the horizontal lines represent the intervals $[\partial_i, \partial_i + 1]$. The numbers above them are labels for the characteristic vectors. That is, the 1st basic net intervals give rise to the following characteristic vectors:

| | | |
|---|---|---|
| $S_1$ | $\langle 1, \ [0], i\rangle$ | $(i = 1, \ldots, m-3)$ |
| $S_2$ | $\left\langle \frac{q-m+2}{3}, \ [0], 1\right\rangle$ | |
| $S_3$ | $\left\langle \frac{-q+m+1}{3}, \ \left[0, \frac{q-m+2}{3}\right], i\right\rangle$ | $(i = 1, 2, 3)$ |
| $S_4$ | $\left\langle \frac{2q-2m+1}{3}, \ \left[\frac{-q+m+1}{3}\right], i\right\rangle$ | $(i = 1, 2)$ |
| $S_5$ | $\left\langle \frac{q-m+2}{3}, \ \left[\frac{-q+m+1}{3}\right], 1\right\rangle$ | |

Note that $S_4$ is not degenerate because its first component is positive:

$$\frac{2q-2m+1}{3}=\frac{\sqrt{m^2-4m+7}-m+2}{3}=\frac{\sqrt{m^2-4m+7}-\sqrt{m^2-4m+4}}{3}>0.$$

Figure 5.1.2 and 5.1.3 illustrate the 2nd iteration. They reveal that all characteristic vectors have already appeared among the 1st basic net intervals. The behavior of $\Phi$ is

$$\begin{aligned}
1 &\mapsto 1^{m-3}\cdot 2\cdot 3^3\cdot 4^2\cdot 5,\\
2 &\mapsto 1^{(m-3)/2}\cdot 2\cdot 3^2\cdot 4^2,\\
3 &\mapsto 1^{(m-5)/2}\cdot 2\cdot 3^2\cdot 5,
\end{aligned}$$

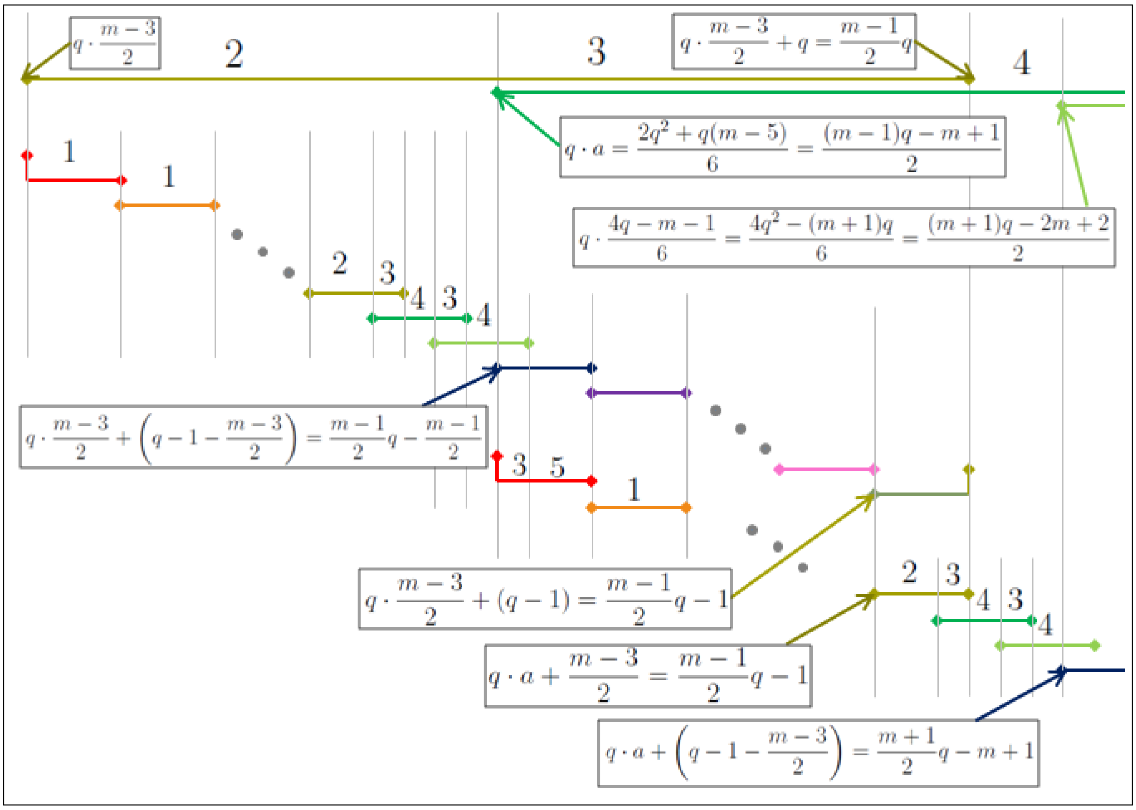



Figure 5.1.2: 2nd iteration of Example I. (part one)

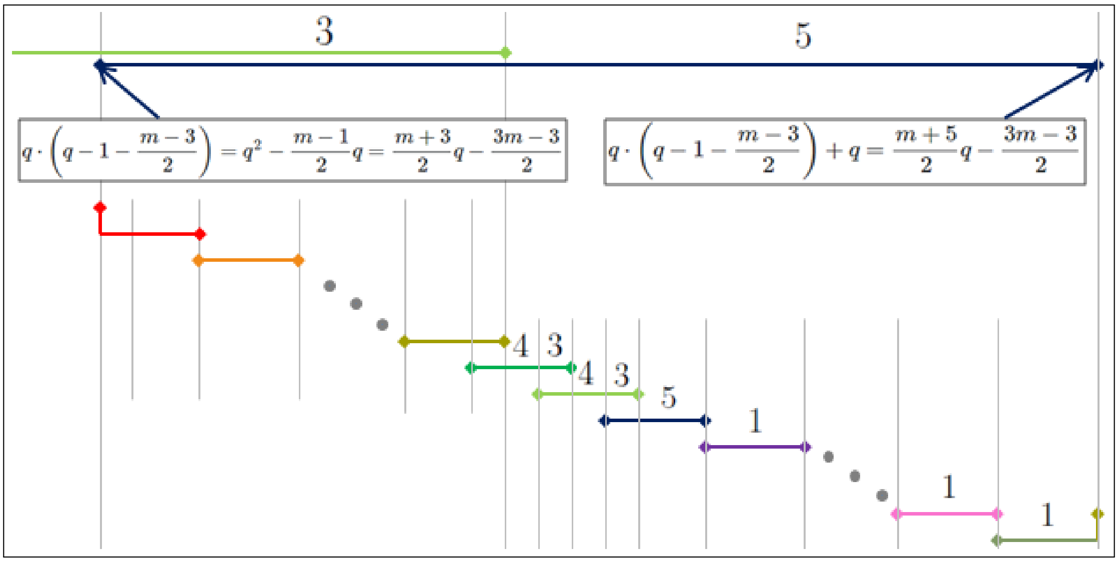



Figure 5.1.3: 2nd iteration of Example I. (part two)

$$4 \mapsto 3 \cdot 4^2,$$
$$5 \mapsto 1^{(m-3)/2} \cdot 3^2 \cdot 4^2 \cdot 5,$$

where for instance the expression "$4 \mapsto 3 \cdot 4^2$" means that an $n$-th basic net interval of type $S_4$ produces three $(n+1)$-th basic net intervals, one of type $S_3$ and two of type $S_4$.

We end by studying some properties of $\Phi$. For small $m$, we have

| $m$ | $\sigma_A(\partial_1)\ \cdots\ \sigma_A(\partial_m)$ | Code of $\mathbb{P}(A)$ |
|---|---|---|
| 5 | $1,\ 0.422,\ \ -0.154,\ \ -0.732,\ 0.267$ | $\langle 125034\rangle$ |
| 7 | $1,\ 2,\ 0.784,\ \ -0.430,\ \ -1.645,\ \ -0.645,\ 0.354$ | $\langle 21370465\rangle$ |
| 9 | $1,\ 2,\ 3,\ 1.131,\ \ -0.737,\ \ -2.605,\ \ -1.605,\ \ -0.605,\ 0.394$ | $\langle 3241908576\rangle$ |

Noting that $A$ tends to 1.5 as $m \to \infty$:

$$\lim_{m\to\infty} \frac{(m+1)-\sqrt{m^2-4m+7}}{2} = \lim_{m\to\infty} \frac{(m+1)^2-(m^2-4m+7)}{2[(m+1)+\sqrt{m^2-4m+7}]} = 1.5,$$

we have asymptotically

| | $\sigma_A(\partial_1)$, | $\cdots$ , | $\sigma_A(\partial_{\frac{m-3}{2}})$, | $\sigma_A(\partial_{\frac{m-1}{2}})$, | $\sigma_A(\partial_{\frac{m+1}{2}})$, | $\sigma_A(\partial_{\frac{m+3}{2}})$, | $\cdots$ , | $\sigma_A(\partial_m)$ |
|---|---|---|---|---|---|---|---|---|
| $\approx$ | 1, | $\cdots$ , | $\frac{m}{2}$, | $\frac{m}{6}$, | $-\frac{m}{6}$, | $-\frac{m}{2}$, | $\cdots$ , | 0.5 |

and $\mathbb{P}(A)$ has code

$$\left\langle \frac{m-3}{2} \cdots \frac{m-1}{2} \cdots 1\ m\ 0\ (m{-}1) \cdots \frac{m+1}{2} \cdots \frac{m+3}{2} \right\rangle .$$

From the figures, we see that

$$Y_{\text{core}} = \{1, \ell_2, \ell_3 + \ell_4\}\,,$$

where $\ell_i$ denotes the first component of $S_i$. Note that

$$\ell_2 = \frac{q-m+2}{3} = \ell_3 + \ell_4$$

and

$$\lim_{m\to\infty} \frac{q-m+2}{3} = \lim_{m\to\infty} \frac{1}{3}\left(\frac{\sqrt{m^2-4m+7}-m}{2} + 2.5\right) = \frac{1}{3}\left(\frac{-4}{2\cdot 2} + 2.5\right) = 0.5.$$

## 5.2 Example II

Let $m \geq 5$ be an odd integer and $a, b \in \mathbb{R}$. Consider the situation

$$\begin{cases} \partial_i = a + (i-1)b & \text{for all } i \in [1,\ (m+1)/2] \\ \partial_i = q - 1 - \partial_{m-i} & \text{for all } i \in [(m+1)/2,\ m], \end{cases}$$

so that we have

$$\begin{array}{c|c|c|c||c|c||c|c|c} \partial_1 & \partial_2 & \cdots & \partial_{(m-1)/2} & \partial_0 & \partial_m & \partial_{(m+1)/2} & \cdots & \partial_{m-1} \\ \hline a & a+b & \cdots & a+b(m-3)/2 & 0 & q-1 & q-1-\partial_{(m-1)/2} & \cdots & q-1-a \end{array} .$$

We make some hypotheses in light of the theorem of $\langle 120534\rangle$ (p.4 Theorem 1.7) and Example I. We assume that $q$ has a real algebraic conjugate $A < -1$, that $\mathbb{P}(A)$ has code

$$\langle 1 \cdots 0 \cdots m \cdots (m{-}1)\rangle\,,$$

and that letting $\tau := \sigma_A(a)$, we have

$$\begin{cases} \sigma_A(b) = \dfrac{A-1}{A} + \dfrac{(A-1)-\tau}{A^2} \\ \sigma_A(a) = \dfrac{(A-1)-\tau}{A}. \end{cases} \tag{5.2}$$

From the last equation, we have $\tau = (A-1)/(A+1)$. Consequently,

$$\begin{aligned} A - 1 &= \sum_{i=1}^{m} \sigma_A([i; i-1]) \\ &= 2\sigma_A(a) + (m-2)\sigma_A(b) \\ &= 2\tau + (m-2)\left(\frac{A-1}{A} + \frac{\tau}{A}\right) \\ &= \frac{2(A-1)}{A+1} + (m-2)\left(\frac{A-1}{A} + \frac{A-1}{A(A+1)}\right). \end{aligned}$$

Multiplying both sides by $A(A+1)/(A-1)$, we have

$$\begin{aligned} &A^2 + A = 2A + (m-2)(A+2) \\ \Rightarrow\ &A^2 - (m-1)A - 2(m-2) = 0, \end{aligned}$$

whence

$$q, A = \frac{(m-1) \pm \sqrt{(m-1)^2 + 8(m-2)}}{2} = \frac{(m-1) \pm \sqrt{m^2 + 6m - 15}}{2}.$$

As $m \geq 5$, we have $(m-1)^2 < (m-1)^2 + 8(m-2)$, so $q$ is positive and $A$ is negative. We have $A < -1$ because

$$\begin{aligned} \frac{(m-1) - \sqrt{m^2+6m-15}}{2} < -1 \quad &\Leftarrow \quad (m+1)^2 < (\sqrt{m^2+6m-15})^2 \\ &\Leftarrow \quad 0 < 4m - 16. \end{aligned}$$

A fortiori

$$q = \left|\frac{m-1}{2}\right| + \left|\frac{\sqrt{(m-1)^2 + 8(m-2)}}{2}\right| > |A| > 1.$$

Note that $q \notin \mathbb{N}$, for otherwise there exists $N \in \mathbb{N}$ such that $N^2 = m^2 + 6m - 15 = (m+3)^2 - 24$, whence

$$((m+3) - N,\ (m+3) + N) = (1, 24) \text{ or } (2, 12) \text{ or } (3, 8) \text{ or } (4, 6).$$

Since $m$ is an integer and $m \geq 5$, this shows a contradiction. From the quadratic equation, we have $(q+1)(q-m) = m-4$, whence $\dfrac{1}{q+1} = \dfrac{q-m}{m-4}$. It follows that

$$a = \frac{q-1}{q+1} = \frac{q^2 - (m+1)q + m}{m-4} = \frac{-2q + 3m - 4}{m-4}.$$

As $q-1 = 2a + (m-2)b$, we also have

$$b = \frac{q-1-2a}{m-2} = \frac{mq-7m+12}{(m-4)(m-2)}.$$

By $q > 1$, we have $a = (q-1)/(q+1) \in (0,1)$, and by (5.2),

$$b = \frac{q-1}{q} + \frac{a}{q} \in \left(\frac{q-1}{q},\ \frac{q-1}{q} + \frac{1}{q}\right) \subseteq (0,1).$$

Therefore, $\Phi = \{\rho x + \partial_i \rho\}_{i=0}^{m}$ is of the required form.

We verify that $\Phi$ satisfies the FTC. The first characteristic vector is:

| | |
|---|---|
| $R_0$ | $\langle 1,\ \ [0]\,, 1\rangle$ |

Figure 5.2.1 illustrates the 1st iteration. The new characteristic vectors are (the 3rd

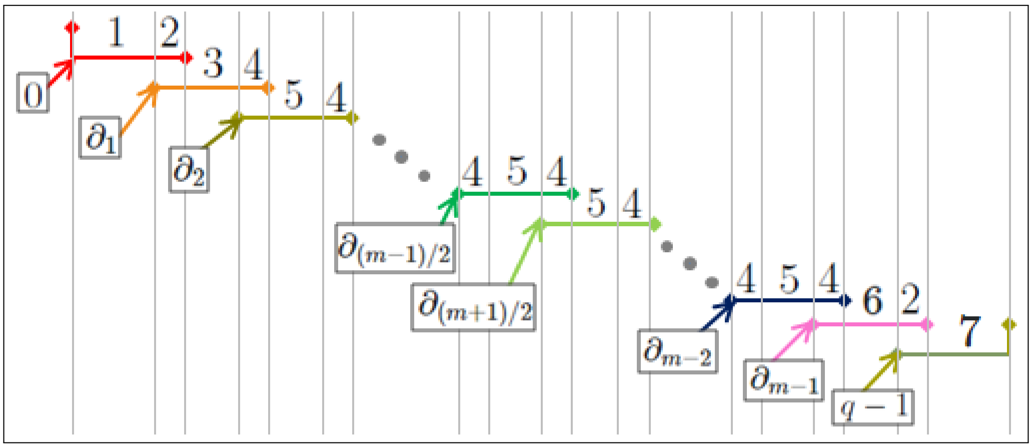



Figure 5.2.1: 1st iteration of Example II.

components are omitted for brevity):

| | | |
|---|---|---|
| $S_1$ | $\langle a,\ \ [0]\rangle$ | $\left\langle \dfrac{-2q+3m-4}{m-4},\ \ [0] \right\rangle$ |
| $S_2$ | $\langle 1-a,\ \ [0,a]\rangle$ | $\left\langle \dfrac{2q-2m}{m-4},\ \ \left[0, \dfrac{-2q+3m-4}{m-4}\right] \right\rangle$ |
| $S_3$ | $\langle \partial_2 - 1,\ \ [1-a]\rangle$ | $\left\langle \dfrac{-q+2m-3}{m-2},\ \ \left[\dfrac{2q-2m}{m-4}\right] \right\rangle$ |
| $S_4$ | $\langle 1-b,\ \ [0,b]\rangle$ | $\left\langle \dfrac{-mq+m^2+m-4}{(m-4)(m-2)},\ \ \left[0, \dfrac{mq-7m+12}{(m-4)(m-2)}\right] \right\rangle$ |
| $S_5$ | $\langle 2b-1,\ \ [1-b]\rangle$ | $\left\langle \dfrac{2mq-m^2-8m+16}{(m-4)(m-2)},\ \ \left[\dfrac{-mq+m^2+m-4}{(m-4)(m-2)}\right] \right\rangle$ |
| $S_6$ | $\langle \partial_2 - 1,\ \ [1-b]\rangle$ | $\left\langle \dfrac{-q+2m-3}{m-2},\ \ \left[\dfrac{-mq+m^2+m-4}{(m-4)(m-2)}\right] \right\rangle$ |
| $S_7$ | $\langle a,\ \ [1-a]\rangle$ | $\left\langle \dfrac{-2q+3m-4}{m-4},\ \ \left[\dfrac{2q-2m}{m-4}\right] \right\rangle$ |

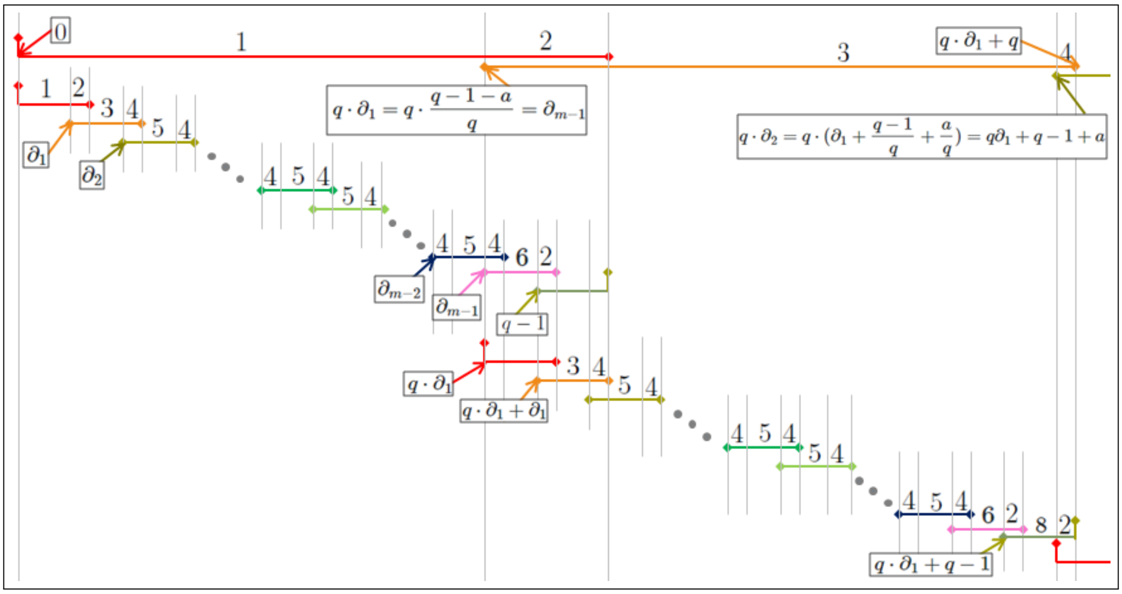



Figure 5.2.2: 2nd iteration of Example II. (part one)

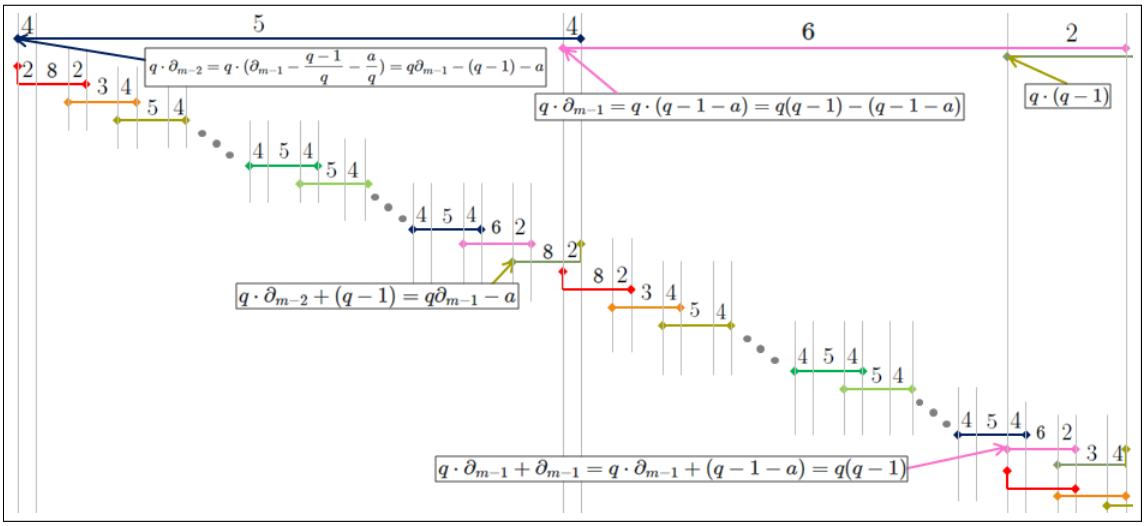



Figure 5.2.3: 2nd iteration of Example II. (part two)

Figure 5.2.2 and 5.2.3 illustrate the 2nd iteration. There is one new characteristic vector:

| | | |
|---|---|---|
| $T_8$ | $\langle 2a-1,\ [1-a] \rangle$ | $\left\langle \dfrac{-4q+5m-4}{m-4},\ \left[\dfrac{2q-2m}{m-4}\right] \right\rangle$ |

We note that $T_8$ is not degenerate because its first component is positive:

$$2 \cdot \frac{q-1}{q+1} = 2a > 1$$

$$\Leftarrow \quad q > 3 \quad \Leftarrow \quad q = \frac{m-1}{2} + \frac{\sqrt{(m-1)^2 + 8(m-2)}}{2} \geq \frac{m-1}{2} + \frac{m-1}{2} \geq 4.$$

Figure 5.2.4 shows that it is the last characteristic vectors.

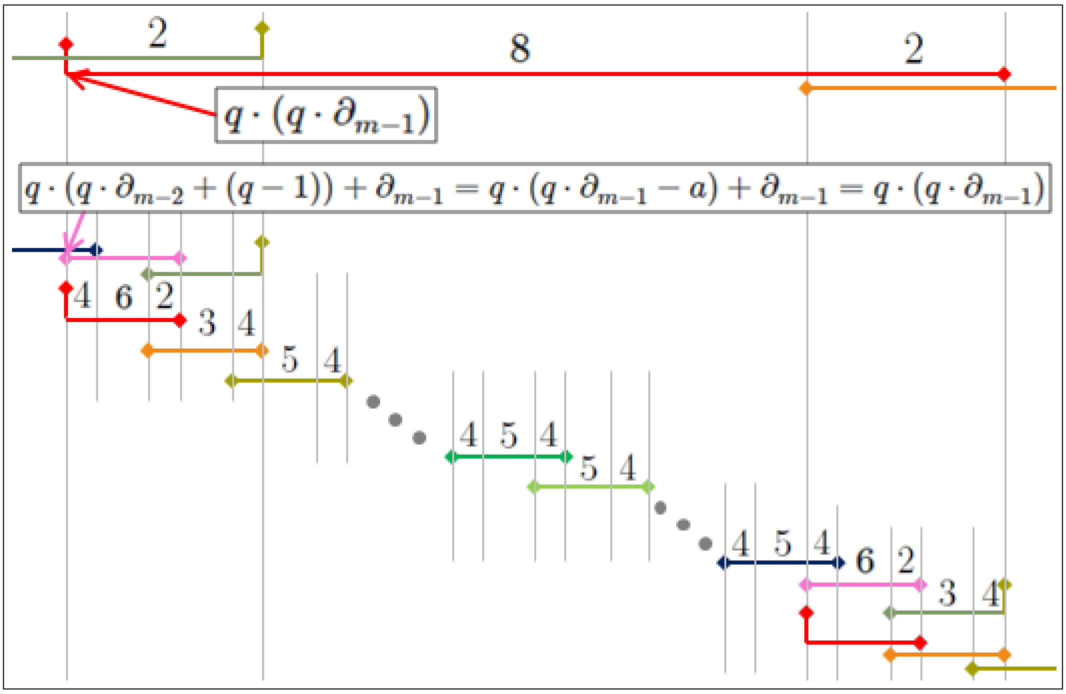



Figure 5.2.4: 3rd iteration of Example II.

The behavior of $\Phi$ is

$$
\begin{aligned}
&0 \mapsto 1 \cdot 2^2 \cdot 3 \cdot 4^{m-2} \cdot 5^{m-3} \cdot 6 \cdot 7,\\
&1 \mapsto 1 \cdot 2 \cdot 3 \cdot 4^{m-3} \cdot 5^{m-3},\\
&2 \mapsto 2 \cdot 3 \cdot 4^2 \cdot 6,\\
&3 \mapsto 2 \cdot 4^{m-3} \cdot 5^{m-3} \cdot 6 \cdot 8,\\
&4 \mapsto 2,\\
&5 \mapsto 2^2 \cdot 3 \cdot 4^{m-2} \cdot 5^{m-3} \cdot 6 \cdot 8^2,\\
&6 \mapsto 2 \cdot 3 \cdot 4^{m-3} \cdot 5^{m-3} \cdot 8,\\
&7 \mapsto 2 \cdot 4^{m-3} \cdot 5^{m-3} \cdot 6 \cdot 7,\\
&8 \mapsto 4^{m-4} \cdot 5^{m-3},
\end{aligned}
$$

where for instance the expression "$2 \mapsto 2 \cdot 3 \cdot 4^2 \cdot 6$" means that an $n$-th basic net interval of type $S_2$ produces five $(n+1)$-th basic net intervals, two of type $S_4$ and three of types $S_2$, $S_3$, $S_6$ respectively.

We can study the properties of $\Phi$ like the previous example. This time we just remark that

$$Y_{\text{core}} = \{a, b\} = \left\{ \frac{q-1-a}{q}, \frac{q^2-1-a}{q^2} \right\},$$

and as $m \to \infty$, we have $A \to -2$:

$$\lim_{m\to\infty} \frac{(m-1) - \sqrt{m^2+6m-15}}{2} = \lim_{m\to\infty} \frac{(m-1)^2 - (m^2+6m-15)}{2[(m-1) + \sqrt{m^2+6m-15}]} = -2,$$

$q \to m$, and $a, b \to 1$.

## 5.3 Example III

Let $m \geq 6$ be an even integer and $a \in \mathbb{R}$. Consider the situation

$$\begin{cases} \partial_i = i & \text{if } i \in [0,\ (m-4)/2] \\ \partial_i = q - 1 - m + i & \text{if } i \in [(m+4)/2,\ m] \end{cases}$$

and

$$\begin{array}{c|c|c|c||c|c|c||c|c|c|c} \partial_{(m-4)/2} & \cdots & \partial_1 & \partial_{(m-2)/2} & \partial_m & \partial_{m/2} & \partial_0 & \partial_{(m+2)/2} & \partial_{m-1} & \cdots & \partial_{(m+4)/2} \\ \hline \dfrac{m-4}{2} & \cdots & 1 & a & q-1 & \dfrac{q-1}{2} & 0 & q-1-a & q-2 & \cdots & q-1-\dfrac{m-4}{2} \end{array}.$$

We assume that $q$ has a real algebraic conjugate $A > 1$, that $\mathbb{P}(A)$ has code

$$\left\langle \frac{m-4}{2}, \cdots, \frac{m-2}{2}, \cdots, m, \cdots, \frac{m}{2}, \cdots, 0, \cdots, \frac{m+2}{2}, \cdots, \frac{m+4}{2} \right\rangle,$$

and that

$$\begin{cases} \sigma_A(\left[\dfrac{m-2}{2}; \dfrac{m-4}{2}\right]) = \dfrac{(A-1) - \frac{m-4}{2}}{A} \\ \sigma_A(\left[\dfrac{m}{2}; \dfrac{m-2}{2}\right]) = \sigma_A(\left[\dfrac{m+2}{2}; \dfrac{m}{2}\right]) = \dfrac{(A-1) - \frac{m-4}{2}}{A} \\ \sigma_A(\left[\dfrac{m+4}{2}; \dfrac{m+2}{2}\right]) = \dfrac{(A-1) - \frac{m-4}{2}}{A}. \end{cases} \tag{5.3}$$

Adding the equations, we have

$$\begin{aligned} A - m + 3 &= 4 \cdot \frac{A - 1 - \frac{m-4}{2}}{A} \\ \Rightarrow A^2 - mA + 3A &= 4A - 2m + 4 \\ \Rightarrow A^2 - (m+1)A + 2m - 4 &= 0, \end{aligned}$$

whence

$$q, A = \frac{(m+1) \pm \sqrt{(m+1)^2 - 4(2m-4)}}{2} = \frac{(m+1) \pm \sqrt{m^2 - 6m + 17}}{2}.$$

As $m \geq 6$, we have $(m+1)^2 \geq (m+1)^2 - 4(2m-4) > 0$, so both $q$ and $A$ are positive real numbers. We have $A > 1$ because

$$\begin{aligned} \frac{(m+1) - \sqrt{m^2 - 6m + 17}}{2} > 1 \quad &\Leftarrow \quad (m-1)^2 > (\sqrt{m^2 - 6m + 17})^2 \\ &\Leftarrow \quad 4m - 16 > 0. \end{aligned}$$

A fortiori $q > A > 1$. Note that $q \notin \mathbb{N}$, for otherwise there exists $N \in \mathbb{N}$ such that $N^2 = m^2 - 6m + 17 = (m-3)^2 + 8$, whence

$$\begin{cases} N - (m-3) = 1 \\ N + (m-3) = 8 \end{cases} \quad \text{or} \quad \begin{cases} N - (m-3) = 2 \\ N + (m-3) = 4. \end{cases}$$

The former gives $2N = 9$, while the latter gives $N = 3$ and $m = 4$. Both are contradictions. From the quadratic equation, we have $q(q-(m+1)) = 4-2m$, whence $q^{-1} = \dfrac{q-(m+1)}{4-2m}$. By (5.3), we have

$$a = \frac{m-4}{2} + \frac{q-1-\frac{m-4}{2}}{q} = \frac{m-2}{2} + \frac{q-(m+1)}{4} = \frac{q}{4} + \frac{m-5}{4},$$

so

$$\partial_{(m+2)/2} = q-1-a = \frac{3q}{4} - \frac{m-1}{4}.$$

Since

$$(q-1) - \frac{m-4}{2} = 1.5 + \frac{\sqrt{m^2-6m+17}}{2} > 0,$$

we see from (5.3) that $\Phi = \{\rho x + \partial_i \rho\}_{i=0}^{m}$ is of the required form.

We verify that $\Phi$ satisfies the FTC. Figure 5.3.1 illustrates the 1st iteration. We

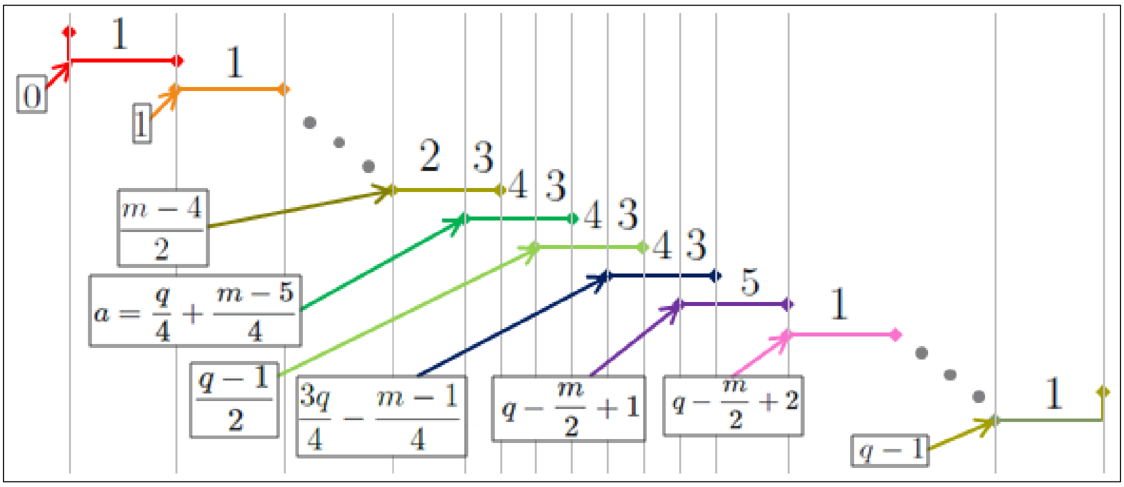



Figure 5.3.1: 1st iteration of Example III.

see that the 1st basic net intervals give rise to the following characteristic vectors:

| | | |
|---|---|---|
| $S_1$ | $\langle 1, \ [0], i \rangle$ | $(i = 1, \ldots, m-4)$ |
| $S_2$ | $\left\langle \frac{q-m+3}{4}, \ [0], 1 \right\rangle$ | |
| $S_3$ | $\left\langle \frac{-q+m+1}{4}, \ \left[0, \frac{q-m+3}{4}\right], i \right\rangle$ | $(i = 1, 2, 3, 4)$ |
| $S_4$ | $\left\langle \frac{q-m+1}{2}, \ \left[\frac{-q+m+1}{4}\right], i \right\rangle$ | $(i = 1, 2, 3)$ |
| $S_5$ | $\left\langle \frac{q-m+3}{4}, \ \left[\frac{-q+m+1}{4}\right], 1 \right\rangle$ | |

$S_4$ is not degenerate because its first component is positive:

$$\frac{q-m+1}{2} = \frac{1}{2}\frac{\sqrt{m^2-6m+17}-m+3}{2} = \frac{\sqrt{m^2-6m+17}-\sqrt{m^2-6m+9}}{4} > 0.$$

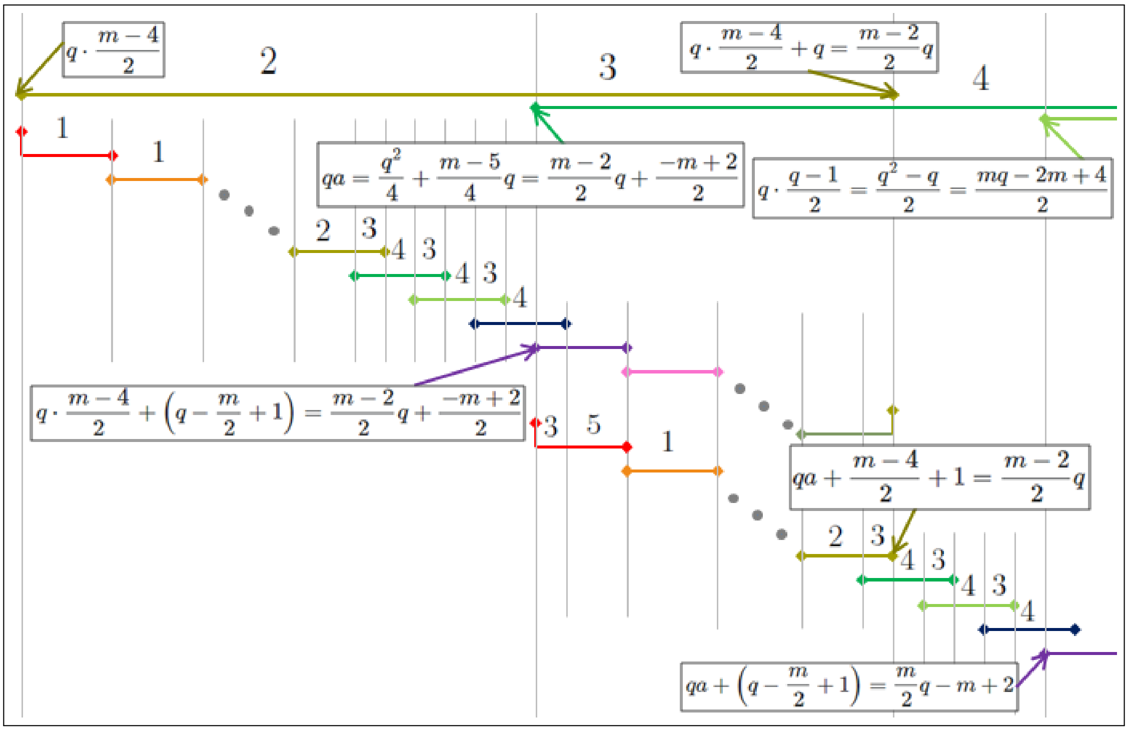



Figure 5.3.2: 2nd iteration of Example III. (part one)

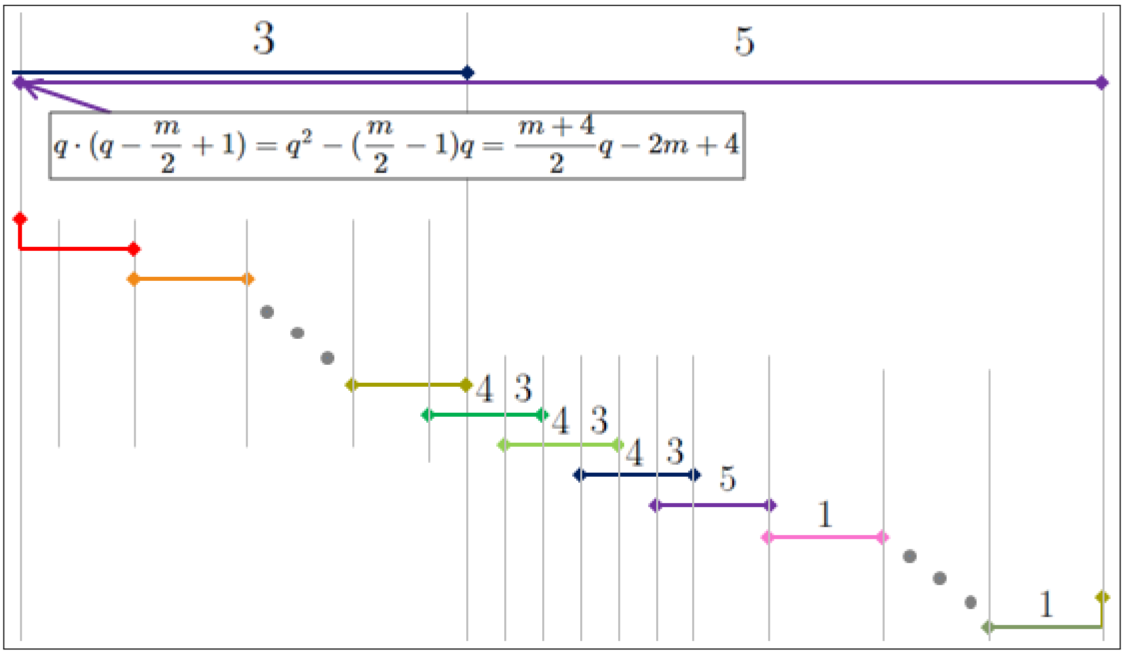



Figure 5.3.3: 2nd iteration of Example III. (part two)

Figure 5.3.2 and 5.3.3 illustrate the 2nd iteration. They reveal that all characteristic vectors have already appeared among the 1st basic net intervals. The behavior of $\Phi$ is

$$1 \mapsto 1^{m-4}\cdot 2\cdot 3^4\cdot 4^3\cdot 5,$$
$$2 \mapsto 1^{(m-4)/2}\cdot 2\cdot 3^3\cdot 4^3,$$

$$\begin{aligned} 3 &\mapsto 1^{(m-6)/2} \cdot 2 \cdot 3^2 \cdot 5, \\ 4 &\mapsto 3^2 \cdot 4^3, \\ 5 &\mapsto 1^{(m-4)/2} \cdot 3^3 \cdot 4^3 \cdot 5, \end{aligned}$$

where for instance the expression "$4 \mapsto 3^2 \cdot 4^3$" means that an $n$-th basic net interval of type $S_4$ produces five $(n+1)$-th basic net intervals, two of type $S_3$ and three of type $S_4$.

We can study the properties of $\Phi$ like Example I. This time we just remark that

$$Y_{\text{core}} = \left\{1,\ \frac{q-m+3}{4}\right\},$$

and $A$ tends to 2 as $m \to \infty$:

$$\lim_{m\to\infty} \frac{(m+1) - \sqrt{m^2 - 6m + 17}}{2} = \lim_{m\to\infty} \frac{(m+1)^2 - (m^2 - 6m + 17)}{2[(m+1) + \sqrt{m^2 - 6m + 17}]} = 2.$$

## 5.4 Example IV

Let $a \in \mathbb{R}$ and consider $\Phi = \{\rho x + \partial_i \rho\}_{i=0}^4$, where

| $\partial_0$ | $\partial_1$ | $\partial_2$ | $\partial_3$ | $\partial_4$ |
|---|---|---|---|---|
| $0$ | $a$ | $(q-1)/2$ | $q-1-a$ | $q-1$ |

.

Solving $A$, $\tau := \sigma_A(a)$ from

$$\left\{ \begin{array}{l} \tau = (A^2 - 1 - \tau)/A^2 \\ (A-1)/2 - \tau = (A^5 - 1 - \tau)/A^5, \end{array} \right. \tag{5.4}$$

we get $\tau = (A^2-1)/(A^2+1)$ and $f(A) = 0$, where $f(x) := x^5 - 4x^4 - 3x^3 - 4x^2 - 4x - 4$. By computer, the roots of $f(x)$ are

$$4.8344222, \qquad 0.2850073 \pm 0.9767886i, \qquad -0.7022184 \pm 0.5532115i,$$

and $|A| = 1.0175191 > 1$ for $A := 0.2850073 + 0.9767886i$. We have the IFS

| $\partial_0$ | $\partial_1$ | $\partial_2$ | $\partial_3$ | $\partial_4$ |
|---|---|---|---|---|
| $0$ | $\dfrac{q^2-1}{q^2+1}$ | $\dfrac{q-1}{2}$ | $q - 1 - \dfrac{q^2-1}{q^2+1}$ | $q-1$ |
| $0$ | $-0.5q^3 + 2q^2 + 2q + 1$ | $0.5q - 0.5$ | $0.5q^3 - 2q^2 - q - 2$ | $q-1$ |
| $0$ | $0.9179374$ | $1.9172111$ | $2.9164848$ | $3.8344222$ |

.

Hence $\Phi$ is of the required form. We verify that it satisfies the FTC. The first characteristic vector is:

| | |
|---|---|
| $R_0$ | $\langle 1,\ [0], 1\rangle$ |

Figure 5.4.1 illustrates the 1st iteration. The new characteristic vectors are (we omit

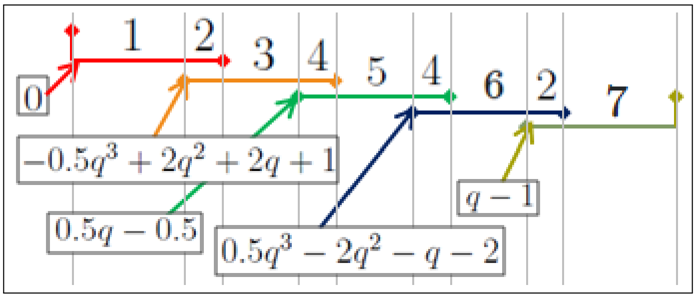



Figure 5.4.1: 1st iteration of Example IV.

the 3rd components for brevity):

| | 1st component | 2nd component |
|---|---:|---:|
| $S_1$ | $-0.5q^3 + 2q^2 + 2q + 1$<br>$= 0.9179374$ | $[0]$ |
| $S_2$ | $0.5q^3 - 2q^2 - 2q$<br>$= 0.0820626$ | $[0, -0.5q^3 + 2q^2 + 2q + 1]$<br>$= [0, 0.9179374]$ |
| $S_3$ | $0.5q - 1.5$<br>$= 0.9172111$ | $[0.5q^3 - 2q^2 - 2q]$<br>$= [0.0820626]$ |
| $S_4$ | $-0.5q^3 + 2q^2 + 1.5q + 2.5$<br>$= 0.0007263$ | $[0, 0.5q^3 - 2q^2 - 1.5q - 1.5]$<br>$= [0, 0.9992737]$ |
| $S_5$ | $q^3 - 4q^2 - 3q - 4$<br>$= 0.9985474$ | $[-0.5q^3 + 2q^2 + 1.5q + 2.5]$<br>$= [0.0007263]$ |
| $S_6$ | $0.5q - 1.5$<br>$= 0.9172111$ | $[-0.5q^3 + 2q^2 + 1.5q + 2.5]$<br>$= [0.0007263]$ |
| $S_7$ | $-0.5q^3 + 2q^2 + 2q + 1$<br>$= 0.9179374$ | $[0.5q^3 - 2q^2 - 2q]$<br>$= [0.0820626]$ |

Figure 5.4.2 and 5.4.3 illustrate the 2nd iteration. The new characteristic vectors

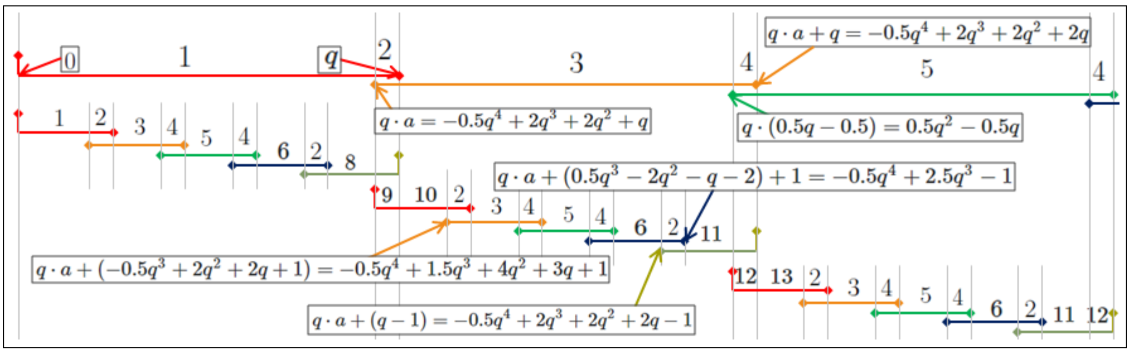



Figure 5.4.2: 2nd iteration of Example IV. (part one)

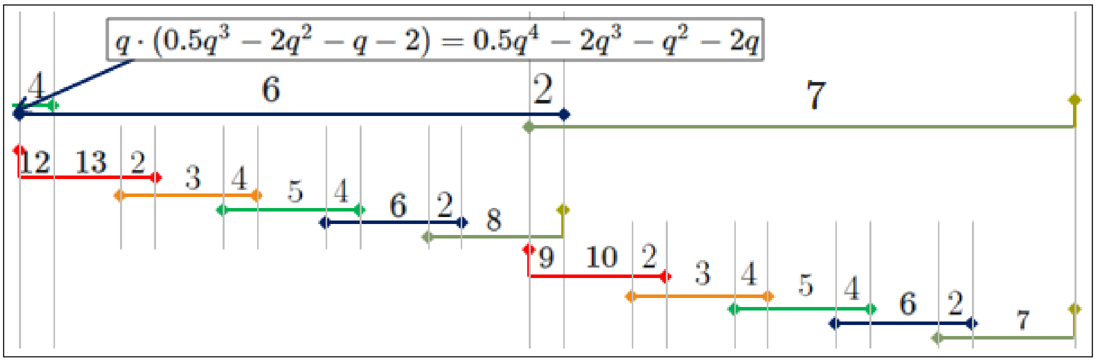



Figure 5.4.3: 2nd iteration of Example IV. (part two)

are:

| | 1st component | 2nd component |
|---|---|---|
| $T_8$ | $-0.5q^4 + 1.5q^3 + 4q^2 + 2q + 1$ $= 0.5212121$ | $[0.5q^3 - 2q^2 - 2q]$ $= [0.0820626]$ |
| $T_9$ | $0.5q^4 - 2q^3 - 2q^2$ $= 0.3967253$ | $[0, -0.5q^4 + 2q^3 + 2q^2 + 1]$ $= [0, 0.6032747]$ |
| $T_{10}$ | $-0.5q^4 + 1.5q^3 + 4q^2 + 2q + 1$ $= 0.5212121$ | $[0.5q^4 - 2q^3 - 2q^2]$ $= [0.3967253]$ |
| $T_{11}$ | $0.5q^4 - 2.5q^3 + 0.5q^2 - 0.5q + 1$ $= 0.9144262$ | $[0.5q^3 - 2q^2 - 2q]$ $= [0.0820626]$ |
| $T_{12}$ | $-0.5q^4 + 2q^3 + 1.5q^2 + 2.5q$ $= 0.0035112$ | $[0, 0.5q^4 - 2q^3 - 1.5q^2 - 2.5q + 1]$ $= [0, 0.9964888]$ |
| $T_{13}$ | $0.5q^4 - 2.5q^3 + 0.5q^2 - 0.5q + 1$ $= 0.9144262$ | $[-0.5q^4 + 2q^3 + 1.5q^2 + 2.5q]$ $= [0.0035112]$ |

Figure 5.4.4 and 5.4.5 show the 3rd iteration. It gives rise to:

| | 1st component | 2nd component |
|---|---|---|
| $U_{14}$ | $-0.5q^3 + 1.5q^2 + 4q + 3$ $= 0.9009628$ | $[0.5q^3 - 2q^2 - 2q]$ $= [0.0820626]$ |
| $U_{15}$ | $0.5q^2 - 2q - 2$ $= 0.0169746$ | $[0, -0.5q^2 + 2q + 3]$ $= [0, 0.9830254]$ |
| $U_{16}$ | $-0.5q^3 + 1.5q^2 + 4q + 3$ $= 0.9009628$ | $[0.5q^2 - 2q - 2]$ $= [0.0169746]$ |

The 4th iteration is illustrated in Figure 5.4.6. There is only one new characteristic vector:

| | 1st component | 2nd component |
|---|---|---|
| $V_{17}$ | $-q^3 + 4q^2 + 4q + 1$ $= 0.8358748$ | $[0.5q^3 - 2q^2 - 2q]$ $= [0.0820626]$ |

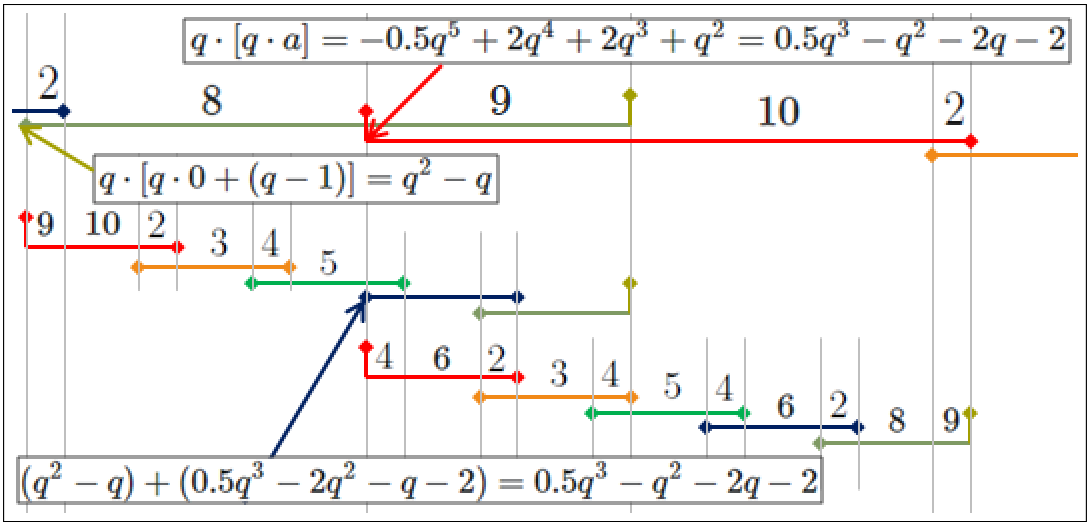



Figure 5.4.4: 3rd iteration of Example IV. (part one)

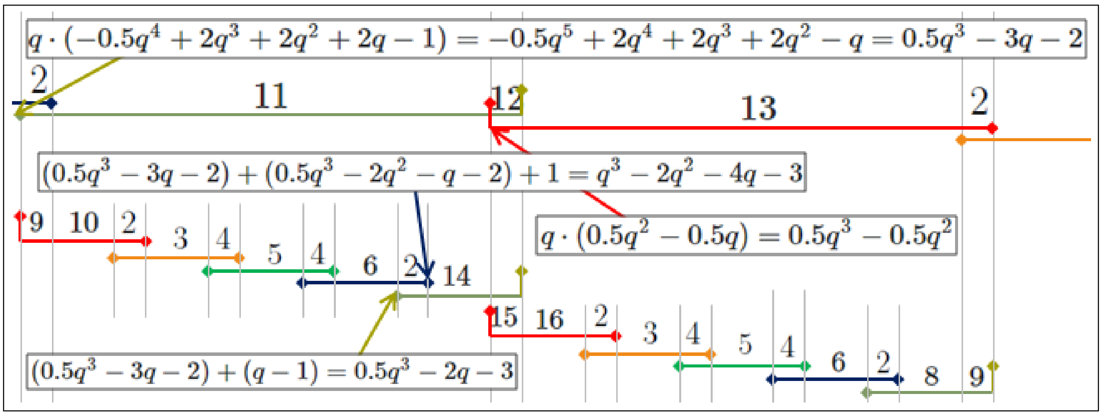



Figure 5.4.5: 3rd iteration of Example IV. (part two)

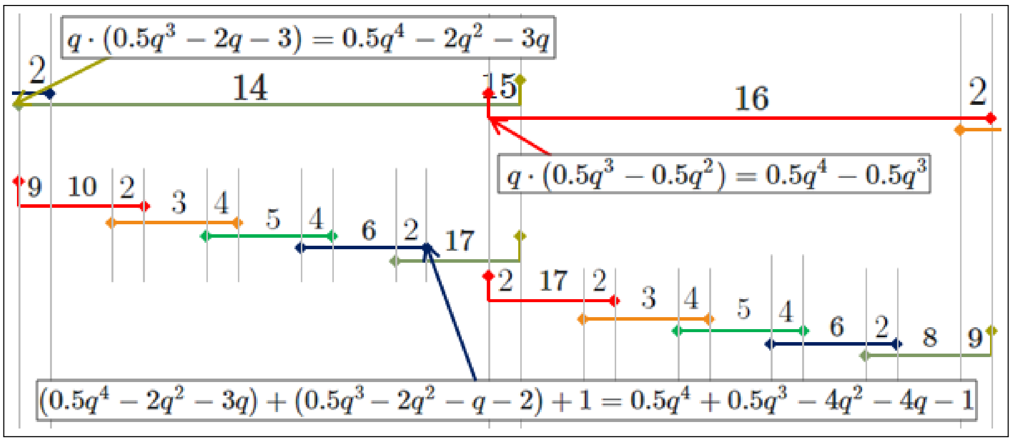



Figure 5.4.6: 4th iteration of Example IV.

As Figure 5.4.7 demonstrates, this is the last characteristic vector.

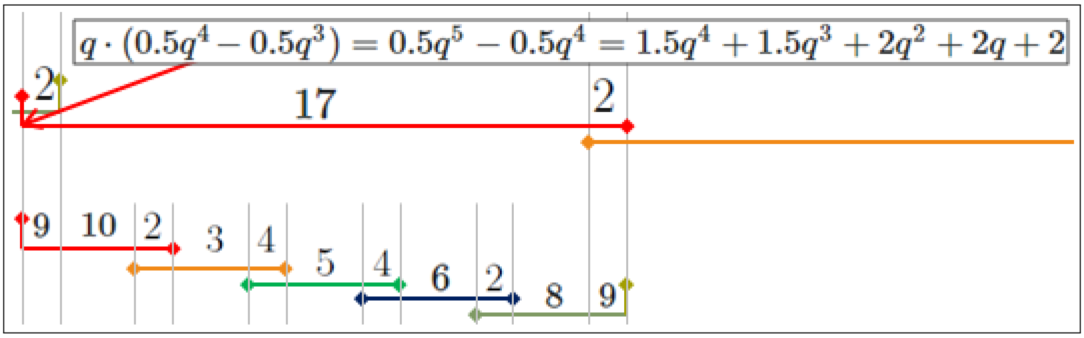



Figure 5.4.7: 5th iteration of Example IV.

As a result, the associated matrix is

$$\left[\begin{array}{c|cccccccccccccccccc}
 & R_0 & S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & T_8 & T_9 & T_{10} & T_{11} & T_{12} & T_{13} & U_{14} & U_{15} & U_{16} & V_{17} \\
\hline
R_0 & 0 & 1 & 11 & 1 & 11 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
S_1 & 0 & 1 & 11 & 1 & 11 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
S_2 & 0 & 0 & 00 & 0 & 00 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 00 & 0 & 00 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
S_3 & 0 & 0 & 11 & 1 & 11 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
S_4 & 0 & 0 & 00 & 0 & 00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 00 & 0 & 00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
S_5 & 0 & 0 & 11 & 1 & 11 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
S_6 & 0 & 0 & 11 & 1 & 11 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
S_7 & 0 & 0 & 11 & 1 & 11 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
T_8 & 0 & 0 & 10 & 1 & 10 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
T_9 & 0 & 0 & 10 & 1 & 11 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
T_{10} & 0 & 0 & 10 & 0 & 10 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
T_{11} & 0 & 0 & 11 & 1 & 11 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
T_{12} & 0 & 0 & 00 & 0 & 00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
T_{13} & 0 & 0 & 11 & 1 & 11 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
U_{14} & 0 & 0 & 11 & 1 & 11 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
U_{15} & 0 & 0 & 10 & 0 & 00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
U_{16} & 0 & 0 & 11 & 1 & 11 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
V_{17} & 0 & 0 & 11 & 1 & 11 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right].$$

Reading from the figures, we find that

$$\begin{aligned} Y_{\text{core}} = \{\ & \ell_1,\ \ell_2+\ell_3,\ \ell_4+\ell_5,\ \ell_4+\ell_6, \\ & \ell_2+\ell_8,\ \ell_9+\ell_{10},\ \ell_2+\ell_{11},\ \ell_{12}+\ell_{13}, \\ & \ell_2+\ell_{14},\ \ell_{15}+\ell_{16},\ \ell_2+\ell_{17}\ \}, \end{aligned}$$

where for instance $\ell_3$ means the first component of $S_3$, $\ell_{17}$ means the first component of $V_{17}$, etc. We proceed to simplify the R.H.S.. From (5.4) and $a = -0.5q^3 + 2q^2 + 2q + 1$, we have

$$\frac{q-1}{2} + 0.5q^3 - 2q^2 - 2q - 1 = \frac{q^5 - 1 - a}{q^5}.$$

This and $q^5 - 4q^4 - 3q^3 - 4q^2 - 4q - 4 = 0$ give

$$\begin{cases} 0.5q^8 - 2q^7 - 1.5q^6 - 2.5q^5 + 1 + a = 0; \\ -0.5q^5 + 2q^4 + 2q^3 + 1 + a = 0. \end{cases}$$

From these, (5.4), and the above tables of the characteristic vectors, we find that indeed

$$Y_{\text{core}} = \left\{ \frac{q^i - 1 - a}{q^i} : i = 1, 2, 3, 4, 5 \right\},$$

because

$$\begin{aligned}
\ell_1 &= -0.5q^3 + 2q^2 + 2q + 1 \\
&= \ell_4 + \ell_6 = \ell_9 + \ell_{10} = \ell_{12} + \ell_{13} = \ell_{15} + \ell_{16} = \ell_2 + \ell_{17}, \\
\ell_1 &= a = \frac{q^2 - 1 - a}{q^2}, \\
\ell_2 + \ell_3 &= 0.5q^3 - 2q^2 - 1.5q - 1.5 \\
&= \ell_4 + \ell_5, \\
\ell_2 + \ell_3 &= \frac{q-1}{2} - a = \frac{q^5 - 1 - a}{q^5}, \\
\ell_2 + \ell_8 &= -0.5q^4 + 2q^3 + 2q^2 + 1 = \frac{q - 1 - a}{q}, \\
\ell_2 + \ell_{11} &= 0.5q^4 - 2q^3 - 1.5q^2 - 2.5q + 1 = \frac{q^4 - 1 - a}{q^4}, \\
\ell_2 + \ell_{14} &= -0.5q^2 + 2q + 3 = \frac{q^3 - 1 - a}{q^3}.
\end{aligned}$$

Finally, we record the codes of $\mathbb{L}_j(A, 1)$ $(0 \leq j \leq 5)$:

| $j$ | $\sigma_A(\partial_1)/A^j$ | $\sigma_A(\partial_2)/A^j$ | $\sigma_A(\partial_3)/A^j$ | $\sigma_A(\partial_4)/A^j$ | Code of $\mathbb{L}_j$ |
|---|---|---|---|---|---|
| 0 | $0.220 + 3.414i$ | $-0.357 + 0.488i$ | $-0.935 - 2.437i$ | $-0.714 + 0.976i$ | $\langle 10243 \rangle$ |
| 1 | $3.281 + 0.731i$ | $0.362 + 0.471i$ | $-2.557 + 0.211i$ | $0.724 + 0.943i$ | $\langle 14203 \rangle$ |
| 2 | $1.593 - 2.894i$ | $0.544 - 0.212i$ | $-0.504 + 2.470i$ | $1.089 - 0.424i$ | $\langle 14203 \rangle$ |
| 3 | $-2.292 - 2.300i$ | $-0.050 - 0.572i$ | $2.192 + 1.155i$ | $-0.100 - 1.144i$ | $\langle 30241 \rangle$ |
| 4 | $-2.801 + 1.529i$ | $-0.553 - 0.110i$ | $1.693 - 1.750i$ | $-1.107 - 0.220i$ | $\langle 30241 \rangle$ |
| 5 | $0.671 + 3.063i$ | $-0.256 + 0.492i$ | $-1.184 - 2.079i$ | $-0.513 + 0.984i$ | $\langle 10243 \rangle$ |

## 5.5 Example V

Let $a \in \mathbb{R}$ and consider $\Phi = \{\rho x + \partial_i \rho\}_{i=0}^4$, where

$$\begin{array}{c|c|c|c|c} \partial_0 & \partial_1 & \partial_2 & \partial_3 & \partial_4 \\ \hline 0 & a & (q-1)/2 & q-1-a & q-1 \end{array}.$$

Solving $A$, $\tau := \sigma_A(a)$ from

$$\left\{ \begin{array}{l} \tau = (A^4 - 1 - \tau)/A^4 \\ (A-1)/2 - \tau = (A^3 - 1 - \tau)/A^3, \end{array} \right. \tag{5.5}$$

we get $\tau = (A^4-1)/(A^4+1)$ and $f(A) = 0$, where $f(x) := x^4 - 4x^3 - 4x^2 - 4x + 1$. By computer, the roots of $f(x)$ are

$$4.9606929, \qquad -0.5811388 \pm 0.8138044i, \qquad 0.2015847.$$

Since $f(1/x) = f(x)/x^4$, we see that the non-real roots $A, \overline{A}$ of $f(x)$ satisfy $|A| = 1$. We have the IFS

| $\partial_0$ | $\partial_1$ | $\partial_2$ | $\partial_3$ | $\partial_4$ |
|---|---|---|---|---|
| $0$ | $\dfrac{q^4-1}{q^4+1}$ | $\dfrac{q-1}{2}$ | $q - 1 - \dfrac{q^4-1}{q^4+1}$ | $q-1$ |
| $0$ | $0.5q^3 - 2.5q^2 + 0.5q - 1$ | $0.5q - 0.5$ | $-0.5q^3 + 2.5q^2 + 0.5q$ | $q-1$ |
| $0$ | $0.9967028$ | $1.9803465$ | $2.9639901$ | $3.9606929$ |

.

Hence $\Phi$ is of the required form. We verify that it satisfies the FTC. The first characteristic vector is:

| $R_0$ | $\langle 1, \ [0], 1 \rangle$ |
|---|---|

Figure 5.5.1 illustrates the 1st iteration. The new characteristic vectors are (we omit

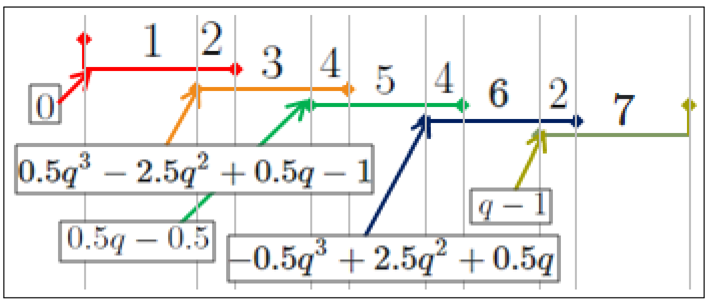



Figure 5.5.1: 1st iteration of Example V.

the 3rd components for brevity):

| | 1st component | 2nd component |
|---|---|---|
| $S_1$ | $0.5q^3 - 2.5q^2 + 0.5q - 1$ $= 0.9967028$ | $[0]$ |
| $S_2$ | $-0.5q^3 + 2.5q^2 - 0.5q + 2$ $= 0.0032972$ | $[0, 0.5q^3 - 2.5q^2 + 0.5q - 1]$ $= [0, 0.9967028]$ |
| $S_3$ | $0.5q - 1.5$ $= 0.9803465$ | $[-0.5q^3 + 2.5q^2 - 0.5q + 2]$ $= [0.0032972]$ |
| $S_4$ | $0.5q^3 - 2.5q^2 + 0.5$ $= 0.0163564$ | $[0, -0.5q^3 + 2.5q^2 + 0.5]$ $= [0, 0.9836436]$ |
| $S_5$ | $-q^3 + 5q^2$ $= 0.9672873$ | $[0.5q^3 - 2.5q^2 + 0.5]$ $= [0.0163564]$ |
| $S_6$ | $0.5q - 1.5$ $= 0.9803465$ | $[0.5q^3 - 2.5q^2 + 0.5]$ $= [0.0163564]$ |
| $S_7$ | $0.5q^3 - 2.5q^2 + 0.5q - 1$ $= 0.9967028$ | $[-0.5q^3 + 2.5q^2 - 0.5q + 2]$ $= [0.0032972]$ |

Figure 5.5.2 and 5.5.3 illustrate the 2nd iteration. The new characteristic vectors

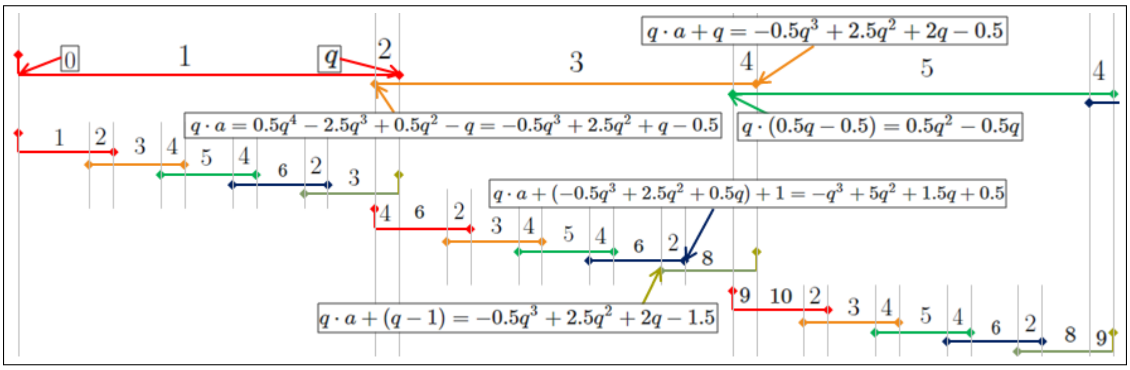



Figure 5.5.2: 2nd iteration of Example V. (part one)

are:

| | 1st component | 2nd component |
|---|---|---|
| $T_8$ | $q^3 - 4.5q^2 - 2q - 0.5$ $= 0.915564$ | $[-0.5q^3 + 2.5q^2 - 0.5q + 2]$ $= [0.0032972]$ |
| $T_9$ | $-0.5q^3 + 2q^2 + 2.5q - 0.5$ $= 0.0811388$ | $[0, 0.5q^3 - 2q^2 - 2.5q + 1.5]$ $= [0, 0.9188612]$ |
| $T_{10}$ | $q^3 - 4.5q^2 - 2q - 0.5$ $= 0.915564$ | $[-0.5q^3 + 2q^2 + 2.5q - 0.5]$ $= [0.0811388]$ |

Figure 5.5.4 shows the 3rd iteration. It gives rise to:

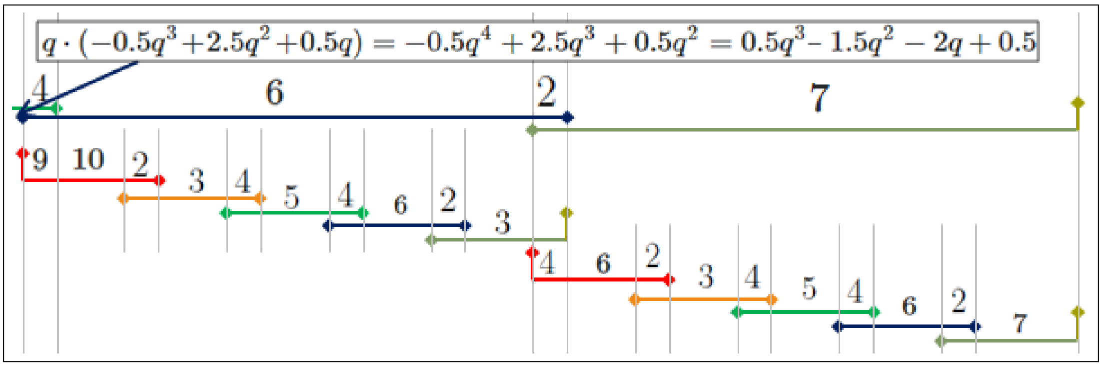



Figure 5.5.3: 2nd iteration of Example V. (part two)

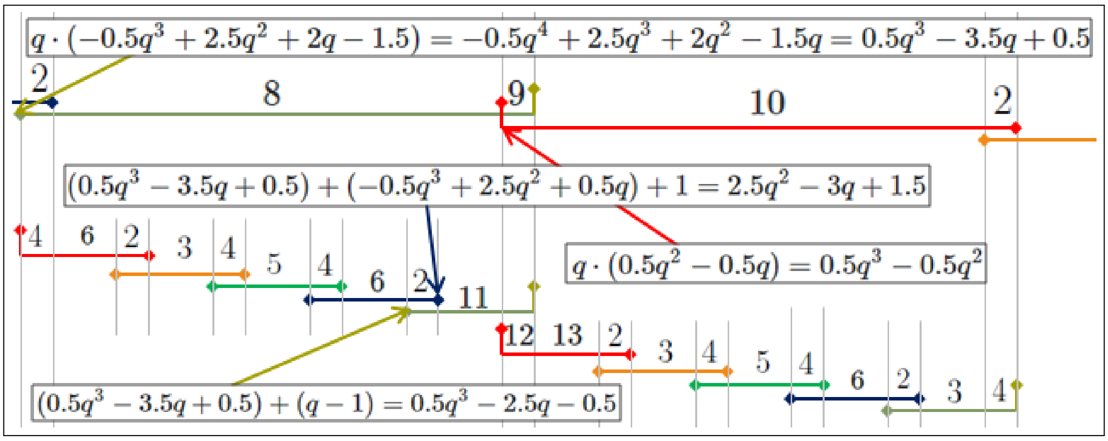



Figure 5.5.4: 3rd iteration of Example V.

| | 1st component | 2nd component |
|---|---|---|
| $U_{11}$ | $0.5q^3-3q^2+3q-1.5$ $= 0.594198$ | $[-0.5q^3+2.5q^2-0.5q+2]$ $= [0.0032972]$ |
| $U_{12}$ | $0.5q^2-2.5q+0.5$ $= 0.4025048$ | $[0, -0.5q^2+2.5q+0.5]$ $= [0, 0.5974952]$ |
| $U_{13}$ | $0.5q^3-3q^2+3q-1.5$ $= 0.594198$ | $[0.5q^2-2.5q+0.5]$ $= [0.4025048]$ |

Figure 5.5.5 demonstrates that they are the last characteristic vectors.

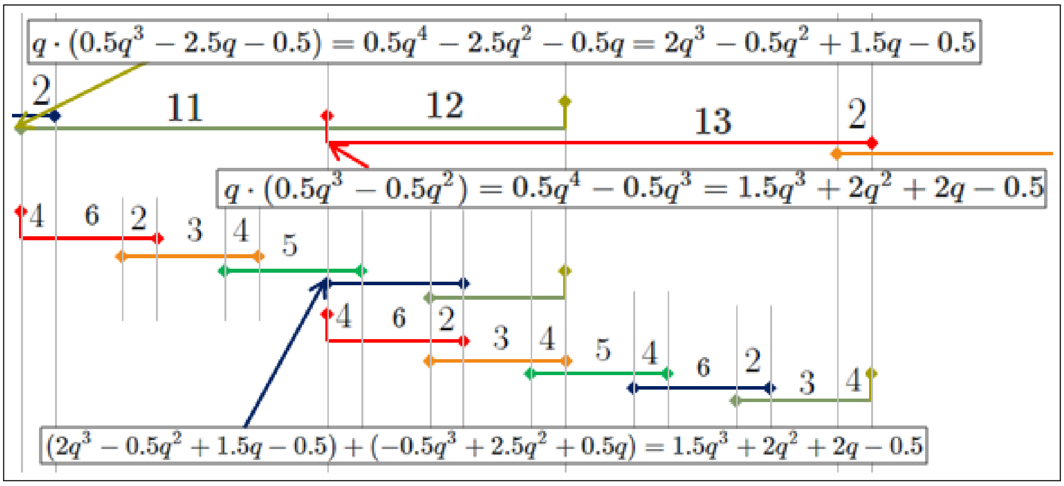



Figure 5.5.5: 4th iteration of Example V.

As a result, the associated matrix is

$$
\begin{array}{c|c}
 & \begin{matrix} R_0 & S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & T_8 & T_9 & T_{10} & U_{11} & U_{12} & U_{13} \end{matrix} \\
\hline
\begin{matrix} R_0 \\ S_1 \\ S_2 \\ \\ S_3 \\ \\ S_4 \\ \\ S_5 \\ S_6 \\ \\ S_7 \\ T_8 \\ T_9 \\ T_{10} \\ U_{11} \\ U_{12} \\ U_{13} \end{matrix} &
\begin{bmatrix}
0 & 1 & 11 & 10 & 11 & 1 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 11 & 11 & 11 & 1 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 00 & 00 & 10 & 0 & 00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 00 & 00 & 10 & 0 & 00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 11 & 10 & 11 & 1 & 11 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 11 & 10 & 11 & 1 & 11 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 00 & 00 & 00 & 0 & 00 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 00 & 00 & 00 & 0 & 00 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 11 & 10 & 11 & 1 & 10 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 11 & 11 & 11 & 1 & 10 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 11 & 11 & 11 & 1 & 10 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 11 & 10 & 11 & 1 & 11 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 11 & 10 & 11 & 1 & 11 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 00 & 00 & 00 & 0 & 00 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 11 & 11 & 11 & 1 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 10 & 10 & 10 & 1 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 10 & 10 & 11 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 10 & 10 & 10 & 1 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}.
$$

Reading from the figures, we find that

$$
\begin{aligned}
Y_{\text{core}} = \{\ & \ell_1,\ \ell_2+\ell_3,\ \ell_4+\ell_5,\ \ell_4+\ell_6, \\
& \ell_2+\ell_8,\ \ell_9+\ell_{10}, \\
& \ell_2+\ell_{11},\ \ell_{12}+\ell_{13}\ \},
\end{aligned}
$$

where for instance $\ell_3$ means the first component of $S_3$, $\ell_8$ means the first component of $T_8$, etc. We proceed to simplify the R.H.S.. From (5.5) and $a = 0.5q^3 - 2.5q^2 + 0.5q - 1$, we have

$$\frac{q-1}{2} - 0.5q^3 + 2.5q^2 - 0.5q + 1 = \frac{q^3 - 1 - a}{q^3}.$$

This and $q^4 - 4q^3 - 4q^2 - 4q + 1 = 0$ give

$$\begin{cases} -0.5q^6 + 2.5q^5 - 0.5q^3 + 1 + a = 0; \\ 0.5q^5 - 2q^4 - 2.5q^3 + 0.5q^2 + 1 + a = 0; \\ -0.5q^3 + 2.5q^2 - 0.5q + 1 + a = 0. \end{cases}$$

From these, (5.5), and the above tables of characteristic vectors, we find that indeed

$$Y_{\text{core}} = \left\{ \frac{q^i - 1 - a}{q^i} : i = 1, 2, 3, 4 \right\},$$

because

$$\begin{aligned} \ell_1 &= 0.5q^3 - 2.5q^2 + 0.5q - 1 \\ &= \ell_4 + \ell_6 = \ell_9 + \ell_{10} = \ell_{12} + \ell_{13}, \\ \ell_1 &= a = \frac{q^4 - 1 - a}{q^4}, \\ \ell_2 + \ell_3 &= -0.5q^3 + 2.5q^2 + 0.5 \\ &= \ell_4 + \ell_5, \\ \ell_2 + \ell_3 &= \frac{q-1}{2} - a = \frac{q^3 - 1 - a}{q^3}, \\ \ell_2 + \ell_8 &= 0.5q^3 - 2q^2 - 2.5q + 1.5 = \frac{q^2 - 1 - a}{q^2}, \\ \ell_2 + \ell_{11} &= -0.5q^2 + 2.5q + 0.5 = \frac{q - 1 - a}{q}. \end{aligned}$$

Finally, taking $A := -0.5811388 + 0.8138044i$, we record the codes of $\mathbb{L}_j(A, 1)$ $(0 \le j \le 4)$:

| $j$ | $\sigma_A(\partial_1)/A^j$ | $\sigma_A(\partial_2)/A^j$ | $\sigma_A(\partial_3)/A^j$ | $\sigma_A(\partial_4)/A^j$ | Code of $\mathbb{L}_j$ |
|---|---|---|---|---|---|
| 0 | $0.000 + 2.914i$ | $-0.790 + 0.406i$ | $-1.581 - 2.100i$ | $-1.581 + 0.813i$ | $\langle 10243 \rangle$ |
| 1 | $2.371 - 1.693i$ | $0.790 + 0.406i$ | $-0.790 + 2.507i$ | $1.581 + 0.813i$ | $\langle 14203 \rangle$ |
| 2 | $-2.756 - 0.945i$ | $-0.128 - 0.879i$ | $2.5 - 0.813i$ | $-0.256 - 1.759i$ | $\langle 30241 \rangle$ |
| 3 | $0.832 + 2.792i$ | $-0.641 + 0.615i$ | $-2.115 - 1.561i$ | $-1.282 + 1.231i$ | $\langle 10243 \rangle$ |
| 4 | $1.789 - 2.300i$ | $0.873 + 0.164i$ | $-0.041 + 2.628i$ | $1.747 + 0.328i$ | $\langle 14203 \rangle$ |

## 5.6 Conclusion

To conclude this chapter, let us first describe how we search for an example for the case $m = 4$. We only present heuristic argument which reflects that we did not know whether there existed such an example.

We think that if such an example exists, then we can find one having the symmetric form

$$\begin{array}{c|c|c|c|c} \partial_0 & \partial_1 & \partial_2 & \partial_3 & \partial_4 \\ \hline 0 & a & (q-1)/2 & q-1-a & q-1 \end{array},$$

because this form should simplify the overlapping behavior of the IFS. It also helps to reduce the number of unknowns: there are only two unknowns $(a, q)$ instead of four $(\partial_1, \partial_2, \partial_3, q)$ in this situation. Observation 2.1 (p.11) is a motivation too.

If $\Phi$ is of this form, then $\partial_4 = 2\partial_2$. Accordingly, $\mathbb{L}_\ell(A, \mathfrak{w})$ always has code $\langle \cdots 4 \cdots 2 \cdots 0 \cdots \rangle$ or $\langle \cdots 0 \cdots 2 \cdots 4 \cdots \rangle$. Reviewing the strategy used in Chapter 4 for the case $m = 3$, we have the following ideas:

1. For a 4-tuple $\Phi$, the codes making the situation intricate were $\langle 1302 \rangle$ and $\langle 1032 \rangle$, and the "compulsory digits" (defined informally in the proof of p.40 Lemma 4.8) were $q - 1 - a$ and $q - 1 - b$. This time, the special codes are $\langle 10243 \rangle$ and $\langle 14203 \rangle$, and the "compulsory digit" is $q - 1 - a$.

2. The proof of Lemma 4.8 (p.40) works, so that $|A| < |1 + \sigma_A(a)|$.

3. The proof of Lemma 4.9 (p.41) works, so that
$$\left| \frac{\sigma_A(a) + 1}{A} \right| \geq |\sigma_A(y) - 1|$$
for all $y \in Y_{\text{core}}$.

4. The proof of Corollary 4.10(ii) (p.42) works, so that $\Re(\sigma_A(a)) \geq 0$.

Letting $\tau := \sigma_A(a)$, we are led to consider the system of equations

$$\left\{ \begin{array}{l} \tau = (A^h - 1 - \tau)/A^h \\ (A - 1)/2 - \tau = (A^k - 1 - \tau)/A^k, \end{array} \right. \tag{5.6}$$

where $(h, k) \in \mathbb{N}^2$, because

1. The first equation gives $\sigma_A(a) = (A^h - 1)/(A^h + 1)$, which fulfills $\Re(\sigma_A(a)) \geq 0$ when $|A| \geq 1$.

2. The previous strategy suggests that given $y \in Y \cap (0, 1)$ such that $\Re(\sigma_A(y)) < 0$, its expansion by our lazy algorithm is forced to use the "compulsory digit" and is of the following form:
$$y = \sum_{i=1}^{k-1} \frac{q-1}{q^i} + \frac{q-1-a}{q^k} + \cdots .$$
If we assume $y = [2; 1]$ and $\dfrac{q-1-a}{q^k}$ is the last non-zero term in the expansion, then we get the second equation of (5.6).

3. Solving (5.6), we get an equation $f(A) = 0$, where $f(x) \in \mathbb{Z}[x]$ is a polynomial whose degree increases when $h, k$ increase. When the degree of $f(x)$ increases, hopefully we can have many algebraic conjugates of $q$, among which some satisfies $|A| \geq 1$.

4. By experiment, or by our assumption that $\Re(\sigma_A(a)) \geq 0 > \Re(\sigma_A([2; 1]))$, it appears that the case $h = k$ is not productive. Having two parameters $(h, k)$ instead one $(h = k)$ increases the flexibility of our model.

We use computer to solve a constrained version of (5.6) and find the promising pairs $(h, k) = (2, 5)$ and $(4, 3)$. Before we proceed to verify the FTC rigorously like the previous section, we also use computer to calculate the characteristic vectors and estimate whether it is likely or not that the FTC holds. Sample codes and output of our computer program are provided in Appendix B and C.

Fortunately, the aforementioned methodology works and we find an example indeed.

* * * * * * * * *

We end by giving miscellaneous results inspired by the examples. To begin with, we observe that for each of them, we have $\#Y_{\text{core}} = \deg q$, the degree of the algebraic number $q$. As the following proposition shows, it suggests that they are, in a certain sense, the simplest overlapping IFS on $\mathbb{R}$ having no hole in their attractors.

**Proposition 5.1 (Lower bound of $\#Y_{\text{core}}$)**
*If $\Phi$ satisfies the FTC, then $\#Y_{\text{core}} \geq \deg q$.*

We do some preparation before giving a proof. As $Y_{\text{core}}$ is a finite set when $\Phi$ satisfies the FTC, we can write $Y_{\text{core}} \stackrel{\text{def}}{=} \{y_1, \ldots, y_H\}$ and define $\mathbf{y} \stackrel{\text{def}}{=} (y_1, \ldots, y_H)$ accordingly. Let

$$X \stackrel{\text{def}}{=} \left\{ \sum_{i=0}^{n-1} q^i d_i : \; d_i \in \mathsf{D}(q), \; n \geq 1 \right\} = \bigcup_{n=1}^{\infty} q^n F_n,$$

where $F_n$ is from p.16 Lemma 2.7. In light of that lemma and noting that $q^n F_n \subseteq q^{n+1} F_{n+1}$ for all $n \geq 1$, we have the following result:

(5.7) If $\Phi$ satisfies the FTC, then for any $s, t \in X$ with $s < t$, there exist a positive integer $\ell \geq 1$ and $\{x_i\}_{i=0}^{\ell} \subseteq X$ such that $s = x_0 < x_1 < \cdots < x_\ell = t$, where $x_i - x_{i-1} \in Y_{\text{core}}$ for all $i \in [1, \ell]$.

**Proof of Proposition 5.1**
For each $y \in Y_{\text{core}} \subseteq Y = X - X$, there exist $x_1, x_2 \in X$ such that $y = x_2 - x_1$. As $qX \subseteq X$ and $\Phi$ satisfies the FTC, using (5.7) with $s := qx_1$, $t := qx_2$, there exist non-negative integers $N_1, \ldots, N_H$ such that

$$qy = N_1 y_1 + \cdots N_H y_H.$$

Therefore, there exists a $H \times H$ non-negative matrix $P$ with integer entries such that $P\mathbf{y} = q\mathbf{y}$. Consequently, $q$ is an eigenvalue of $P$, and $\deg q$ is not greater than the degree of the characteristic polynomial of $P$, which is equal to $H = \#Y_{\text{core}}$.
**Q.E.D.**

We use our examples to illustrate the idea. Recall that in Example IV, $Y_{\text{core}} = \{y_i\}_{i=1}^5$, where $y_i := (q^i - 1 - a)/q^i$ and $a := (q^2 - 1)/(q^2 + 1)$. Using the calculation of $Y_{\text{core}}$ in the last part of that example, and reading from p.64 Figure 5.4.2, we have

$$\begin{aligned} qy_5 = q(\ell_2 + \ell_3) &= (\ell_9 + \ell_{10}) + (\ell_2 + \ell_3) + (\ell_4 + \ell_5) + (\ell_4 + \ell_6) + (\ell_2 + \ell_{11}) \\ &= 2y_2 + y_4 + 2y_5. \end{aligned}$$

We obtain the equations for $qy_i$ $(1 \leq i \leq 4)$ in the same way. Altogether, we have

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 2 \\ 1 & 2 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 & 2 \\ 0 & 2 & 1 & 0 & 2 \\ 0 & 2 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = q \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix}.$$

Similarly, in the case of Example I, we have

$$\begin{pmatrix} 3 & (m-3)/2 \\ 3 & m-2 \end{pmatrix} \begin{pmatrix} y_1 \\ 1 \end{pmatrix} = q \begin{pmatrix} y_1 \\ 1 \end{pmatrix},$$

where $y_1 := (q-m+2)/3$.

Next, we observe that many of our examples not only satisfy the FTC but also the following condition:

(5.8) Given any $y \in Y_{\mathrm{core}}$, we have $|qy - (q-1)| \in Y_{\mathrm{core}}$.

Note that if $qy = N_1 y_1 + \cdots N_H y_H$, $q-1 = L_1 y_1 + \cdots + L_H y_H$, and $|qy-(q-1)| = y_i$, then letting

$$\mathbf{u} := (u_1 \ \cdots \ u_H) := \begin{cases} (L_1 \ \cdots \ L_H) & \text{if } qy > q-1 \\ (N_1 \ \cdots \ (N_i{+}1) \ \cdots \ N_H) & \text{otherwise,} \end{cases}$$

we have $\mathbf{u} \cdot \mathbf{y} = q-1$ and

$$qy = \begin{cases} \mathbf{u} \cdot \mathbf{y} + y_i & \text{if } qy > q-1 \\ \mathbf{u} \cdot \mathbf{y} - y_i & \text{otherwise.} \end{cases}$$

Therefore, under condition (5.8), we can require that the matrix $P$ in the preceding proof is of the form

$$P = \begin{pmatrix} u_{1,1} & \cdots & u_{1,H} \\ \vdots & \vdots & \vdots \\ u_{H,1} & \cdots & u_{H,H} \end{pmatrix} + S,$$

where $u_{i,j}$ are non-negative integers such that

$$(u_{i,1} \ \cdots \ u_{i,H}) \cdot \mathbf{y} = q-1$$

for all $i \in [1, H]$, and that the matrix $S$ satisfies the following property:

(5.9) Each row of $S$ has exactly one non-zero entry which is eqaul to $\pm 1$. Consequently, for each $n \in \mathbb{N}$ the same holds for the matrix $S^n$.

For instance, in the case of Example IV, we have $q-1 = 2y_2 + 2y_5$ and

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 2 \\ 1 & 2 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 & 2 \\ 0 & 2 & 1 & 0 & 2 \\ 0 & 2 & 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Similarly, in the case of Example I with $m=5$ and $y_1 = q/3-1$, we have $q-1 = 3\cdot y_1 + 2\cdot 1$ and

$$P = \begin{pmatrix} 3 & 1 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}.$$

We have the following proposition.

**Proposition 5.2 (Expansion for $Y_{\text{core}}$)**

*If $\Phi$ satisfies the FTC and condition (5.8), then for all $y \in Y_{\text{core}}$, there exists $\{s_i\}_{i=1}^{\infty} \subseteq \{\pm(q-1)\}$ such that*

$$y = \sum_{1}^{\infty} \frac{s_i}{q^i}, \tag{5.10}$$

*where $s_1 = q-1$ and $\{s_i\}_{i=1}^{\infty}$ is eventually periodic. Moreover, we can have uniform pre-period and period.*

**Proof**

Let $\mathbf{v_0}$ be the $H\times 1$ vector with all entries equal to $q-1$. The above discussion gives $q\mathbf{y} = P\mathbf{y} = \mathbf{v_0} + S\mathbf{y}$, that is, $\mathbf{y} = q^{-1}\mathbf{v_0} + (q^{-1}S)\mathbf{y}$. As a result,

$$\mathbf{y} = q^{-1}\mathbf{v_0} + (q^{-1}S)[q^{-1}\mathbf{v_0} + (q^{-1}S)\mathbf{y}] = q^{-1}\mathbf{v_0} + q^{-2}S\mathbf{v_0} + (q^{-1}S)^2\mathbf{y} = \cdots.$$

As a consequence of property (5.9), we have the convergent series

$$\mathbf{y} = \sum_{i=1}^{\infty} \frac{S^{i-1}\mathbf{v_0}}{q^i}.$$

Moreover, since there are only finitely many $H\times H$ matrices satisfying property (5.9), there exist $i \neq j$ such that $S^i = S^j$. This implies that the expansion obtained is eventually periodic. As $\mathbf{y}$ consists of all elements of $Y_{\text{core}}$, we have uniform pre-period and period.

**Q.E.D.**

For instance, in the case of Example IV, we have $S^3 = S^7$ and

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = (q-1)\begin{pmatrix} \mathtt{1\bar{1}\bar{1}(11\bar{1}\bar{1})}^{\omega} \\ \mathtt{11\bar{1}(\bar{1}11\bar{1})}^{\omega} \\ \mathtt{111(\bar{1}\bar{1}11)}^{\omega} \\ \mathtt{111(1\bar{1}\bar{1}1)}^{\omega} \\ \mathtt{111(11\bar{1}\bar{1})}^{\omega} \end{pmatrix},$$

where $\bar{1} := -1$ and

$$\mathtt{c_1c_2c_3(c_4c_5c_6c_7)}^{\omega} := \frac{c_1}{q} + \frac{c_2}{q^2} + \frac{c_3}{q^3} + \left(\frac{c_4}{q^4} + \frac{c_5}{q^5} + \frac{c_6}{q^6} + \frac{c_7}{q^7}\right)\left(1 + \frac{1}{q^4} + \frac{1}{q^8} + \frac{1}{q^{12}} + \cdots\right).$$

Alternatively, since $S^3 = -S^5$, we also have e.g.

$$y_1 = \frac{q-1}{q} + \frac{-(q-1)}{q^2} + \frac{-(q-1)}{q^3} + \left(\frac{q-1}{q^4} + \frac{q-1}{q^5}\right)\left(1 - \frac{1}{q^2} + \frac{1}{q^4} - \frac{1}{q^6} + \cdots\right).$$

We finish by giving two corollaries of the preceding proposition. Firstly, given an algebraic conjugate $A$ of $q$ with $|A| > 1$, we can obtain a simple upper bound of the set $\sigma_A(Y_{\text{core}})$ in terms of $A$. For, given $y \in Y_{\text{core}}$ and expansion (5.10), by using the FTC (p.11 Proposition 2.2) or by using periodicity to write the expansion as a finite sum, we have

$$|\sigma_A(y)| = \left|\sum_1^\infty \frac{\sigma_A(s_i)}{A^i}\right| = \left|\sum_1^\infty \frac{\pm(A-1)}{A^i}\right| \leq \sum_1^\infty \frac{|A-1|}{|A|^i} = \frac{|A-1|}{|A|-1}.$$

This bound is sharp in the case of Example I and II with $m = 5$. In the former case, we have $q^2 - 6q + 6 = 0$, $q = 3 + \sqrt{3}$, $A = 3 - \sqrt{3}$,

$$\mathbf{y} = \begin{pmatrix} q/3 - 1 \\ 1 \end{pmatrix} = (q-1)\begin{pmatrix} \mathtt{1}(\bar{\mathtt{1}})^\omega \\ \mathtt{1}(\mathtt{1})^\omega \end{pmatrix},$$

and $|A-1|/(|A|-1) = 1 = y_2$. In the latter case, we have $q^2 - 4q - 6 = 0$, $q = 2 + \sqrt{10}$, $A = 2 - \sqrt{10}$, and

$$\mathbf{y} = \begin{pmatrix} -2q + 11 \\ (5q - 23)/3 \end{pmatrix} = (q-1)\begin{pmatrix} \mathtt{1}(\bar{\mathtt{1}}\mathtt{1})^\omega \\ \mathtt{1}(\mathtt{1}\bar{\mathtt{1}})^\omega \end{pmatrix}.$$

As $\mathtt{1}(\bar{\mathtt{1}}\mathtt{1})^\omega$ matches the sign change of $A^i$, we have $\sigma_A(y_1) = |A-1|/(|A|-1)$.

The second corollary of Proposition 5.2 is that, since $Y_{\text{core}}$ is the building block of the set $X$ (property (5.7)), we can represent the set $(q-1)^{-1}X$ in terms of expansions in $q$ in a structured way. We use Example IV to illustrate the idea. Table 5.1 lists the elements of $(q-1)^{-1}X$ up to $(q-1)^{-1}(q^2-1)$. In the table, the column of $x - x^\downarrow$ is obtained from Figure 5.4.2 and 5.4.3. "rep$(x) = (n_1, n_2, n_3, n_4, n_5)$" means that $x = \sum_{i=1}^5 n_i y_i$. In the last column, we take

$$\mathtt{z}_{-1}\mathtt{z}_0.\mathtt{z}_1\mathtt{z}_2\mathtt{z}_3(\mathtt{z}_4\mathtt{z}_5)^\varpi := \sum_{i=-1}^3 \frac{z_i}{q^i} + \left(\frac{z_4}{q^4} + \frac{z_5}{q^5}\right)\left(1 - \frac{1}{q^2} + \frac{1}{q^4} - \frac{1}{q^6} + \cdots\right),$$

$\mathtt{a} := 10, \mathtt{b} := 11, \mathtt{c} := 12$, etc, and let $\bar{\mathtt{x}} := -x$ for any $x$. For instance, we have $13.103030 = y_1 + 6y_2 + y_4 + 6y_5$, and $(q-1)^{-1}13.103030 = 3.417211$ has expansions

$$\frac{14}{q} + \frac{12}{q^2} + \frac{0}{q^3} + \left(\frac{2}{q^4} + \frac{12}{q^5}\right)\left(1 - \frac{1}{q^2} + \frac{1}{q^4} - \frac{1}{q^6} + \cdots\right)$$

and

$$3 + \frac{2}{q} + \frac{0}{q^2} + \frac{0}{q^3} + \left(\frac{2}{q^4} + \frac{0}{q^5}\right)\left(1 - \frac{1}{q^2} + \frac{1}{q^4} - \frac{1}{q^6} + \cdots\right).$$

Multiple expansions are obtained through relations like $.\mathtt{440}(\mathtt{04})^\varpi = \mathtt{1.000}(\mathtt{00})^\varpi$. For instance, we have

$$4.437697 = .\mathtt{ig2}(\mathtt{4e})^\varpi = \mathtt{3.642}(\mathtt{42})^\varpi = \mathtt{4.202}(\mathtt{4}\bar{\mathtt{2}})^\varpi.$$

We may compare these expansions to the decimal expansions of $n/9$, $n \in \mathbb{N}$.

| No. | $x \in X$ | $x - x^{\downarrow}$ | $\mathrm{rep}(x)$ | $(q-1)^{-1}x$ |
|---|---|---|---|---|
| 0 | 0 | $\infty$ | $(0,0,0,0,0)$ | $0 = \mathtt{.000(00)}^{\varpi}$ |
| 1 | 0.917937 | $\ell_1$ | $(0,1,0,0,0)$ | $0.239393 = \mathtt{.11\bar{1}(\bar{1}1)}^{\varpi}$ |
| 2 | 1.917211 | $\ell_2 + \ell_3$ | $(0,1,0,0,1)$ | $0.500000 = \mathtt{.220(02)}^{\varpi}$ |
| 3 | 2.916484 | $\ell_4 + \ell_5$ | $(0,1,0,0,2)$ | $0.760606 = \mathtt{.331(13)}^{\varpi}$ |
| 4 | 3.834422 | $\ell_4 + \ell_6$ | $(0,2,0,0,2)$ | $1.000000 = \mathtt{.440(04)}^{\varpi} = \mathtt{1.000(00)}^{\varpi}$ |
| 5 | 4.437697 | $\ell_2 + \ell_8$ | $(1,2,0,0,2)$ | $1.157331 = \mathtt{.53\bar{1}(15)}^{\varpi} = \mathtt{1.1\bar{1}\bar{1}(11)}^{\varpi}$ |
| 6 | 5.355634 | $\ell_9 + \ell_{10}$ | $(1,3,0,0,2)$ | $1.396725 = \mathtt{.64\bar{2}(06)}^{\varpi} = \mathtt{1.20\bar{2}(02)}^{\varpi}$ |
| 7 | 6.354908 | $\ell_2 + \ell_3$ | $(1,3,0,0,3)$ | $1.657331 = \mathtt{.75\bar{1}(17)}^{\varpi} = \mathtt{1.31\bar{1}(13)}^{\varpi}$ |
| 8 | 7.354181 | $\ell_4 + \ell_5$ | $(1,3,0,0,4)$ | $1.917937 = \mathtt{.860(28)}^{\varpi} = \mathtt{1.420(24)}^{\varpi}$ |
| 9 | 8.272119 | $\ell_4 + \ell_6$ | $(1,4,0,0,4)$ | $2.157331 = \mathtt{.97\bar{1}(19)}^{\varpi} = \mathtt{2.1\bar{1}\bar{1}(11)}^{\varpi}$ |
| 10 | 9.268608 | $\ell_2 + \ell_{11}$ | $(1,4,0,1,4)$ | $2.417211 = \mathtt{.a80(28)}^{\varpi} = \mathtt{2.200(20)}^{\varpi}$ |
| 11 | 10.186545 | $\ell_{12} + \ell_{13}$ | $(1,5,0,1,4)$ | $2.656605 = \mathtt{.b9\bar{1}(19)}^{\varpi} = \mathtt{2.31\bar{1}(11)}^{\varpi}$ |
| 12 | 11.185819 | $\ell_2 + \ell_3$ | $(1,5,0,1,5)$ | $2.917211 = \mathtt{.ca0(2a)}^{\varpi} = \mathtt{2.420(22)}^{\varpi}$ |
| 13 | 12.185093 | $\ell_4 + \ell_5$ | $(1,5,0,1,6)$ | $3.177817 = \mathtt{.db1(3b)}^{\varpi} = \mathtt{2.531(33)}^{\varpi}$ |
| 14 | 13.103030 | $\ell_4 + \ell_6$ | $(1,6,0,1,6)$ | $3.417211 = \mathtt{.ec0(2c)}^{\varpi} = \mathtt{3.200(20)}^{\varpi}$ |
| 15 | 14.099519 | $\ell_2 + \ell_{11}$ | $(1,6,0,2,6)$ | $3.677090 = \mathtt{.fd1(3b)}^{\varpi} = \mathtt{3.311(3\bar{1})}^{\varpi}$ |
| 16 | 15.017456 | $\ell_{12} + \ell_{13}$ | $(1,7,0,2,6)$ | $3.916484 = \mathtt{.ge0(2c)}^{\varpi} = \mathtt{3.420(20)}^{\varpi}$ |
| 17 | 16.016730 | $\ell_2 + \ell_3$ | $(1,7,0,2,7)$ | $4.177090 = \mathtt{.hf1(3d)}^{\varpi} = \mathtt{3.531(31)}^{\varpi}$ |
| 18 | 17.016004 | $\ell_4 + \ell_5$ | $(1,7,0,2,8)$ | $4.437697 = \mathtt{.ig2(4e)}^{\varpi} = \mathtt{3.642(42)}^{\varpi}$ |
| 19 | 17.933941 | $\ell_4 + \ell_6$ | $(1,8,0,2,8)$ | $4.677090 = \mathtt{.jh1(3f)}^{\varpi} = \mathtt{4.311(3\bar{1})}^{\varpi}$ |
| 20 | 18.537216 | $\ell_2 + \ell_8$ | $(2,8,0,2,8)$ | $4.834422 = \mathtt{.kg0(4g)}^{\varpi} = \mathtt{10.000(00)}^{\varpi}$ |
| 21 | 19.455153 | $\ell_9 + \ell_{10}$ | $(2,9,0,2,8)$ | $5.073816 = \mathtt{.lh\bar{1}(3h)}^{\varpi} = \mathtt{10.11\bar{1}(\bar{1}1)}^{\varpi}$ |
| 22 | 20.454427 | $\ell_2 + \ell_3$ | $(2,9,0,2,9)$ | $5.334422 = \mathtt{.mi0(4i)}^{\varpi} = \mathtt{10.220(02)}^{\varpi}$ |
| 23 | 21.453701 | $\ell_4 + \ell_5$ | $(2,9,0,2,10)$ | $5.595028 = \mathtt{.nj1(5j)}^{\varpi} = \mathtt{10.331(13)}^{\varpi}$ |
| 24 | 22.371638 | $\ell_4 + \ell_6$ | $(2,10,0,2,10)$ | $5.834422 = \mathtt{.ok0(4k)}^{\varpi} = \mathtt{11.000(00)}^{\varpi}$ |

Table 5.1: Representing the set $X$ and $(q-1)^{-1}X$ of Example IV.

# Appendix A

# Finite type condition, algebraic numbers, and density

This appendix has two purposes. The first is to introduce terminology concerning the FTC. This part is based on [20] and [21, section 6]. The second is to give a proof of Theorem 1.1 (p.3) and Theorem 1.9 (p.5) respectively.

Recall that the IFS in consideration is $\{\varphi_i(x) = \rho x + b_i\}_{i=0}^m$, where $0 = b_0 < b_1 < \cdots < b_m = 1 - \rho$ and $b_{i+1} - b_i \leq \rho$ for all $i$. Write $\mathcal{A} \stackrel{\text{def}}{=} \{0, \ldots, m\}$ and let $\mathcal{A}_n$ be the collection of all words $j_1 \cdots j_n$ of length $n$ over $\mathcal{A}$. Given $J := j_1 \cdots j_n \in \mathcal{A}_n$, we write $\varphi_J := \varphi_{j_1} \circ \cdots \circ \varphi_{j_n}$. If $n = 0$, then $\varphi_J$ is just the identity function. Let $P_n$ be the collection of all end points given by the intervals $\varphi_J[0,1]$ with $J \in \mathcal{A}_n$. i.e.

$$P_n \stackrel{\text{def}}{=} \{\varphi_J(0) : J \in \mathcal{A}_n\} \cup \{\varphi_J(1) : J \in \mathcal{A}_n\}.$$

Since $\varphi_0(0) = 0$ and $\varphi_m(1) = 1$, we have $P_n \subseteq P_{n+1}$ for all $n \geq 0$.

Let $\mathcal{F}_n \stackrel{\text{def}}{=} \left\{[e^{(\downarrow P_n)}, e] : e \in P_n^{\times\min}\right\}$. Each element $\Delta \in \mathcal{F}_n$ is called an ***n*-th basic net interval**. These intervals satisfy the following properties: (i) $\bigcup_{\Delta \in \mathcal{F}_n} \Delta = [0,1]$ for any $n \geq 0$; (ii) for any $\Delta_1, \Delta_2 \in \mathcal{F}_n$, either $\Delta_1 = \Delta_2$ or $\operatorname{int}(\Delta_1) \cap \operatorname{int}(\Delta_2) = \emptyset$; (iii) for any $\Delta \in \mathcal{F}_n$ ($n \geq 1$), there is a unique $\widehat{\Delta} \in \mathcal{F}_{n-1}$ such that $\widehat{\Delta} \supseteq \Delta$.

For each $\Delta \in \mathcal{F}_n$, we define a positive number $\ell_n(\Delta)$, a vector $V_n(\Delta)$, and a positive integer $r_n(\Delta)$. Given $\Delta =: [a,b] \in \mathcal{F}_n$, we define

$$\ell_n(\Delta) \stackrel{\text{def}}{=} q^n(b-a)$$

and

$$V_n(\Delta) \stackrel{\text{def}}{=} [a_1, \ldots, a_k],$$

where $a_1 < \cdots < a_k$ are all elements of the set

$$\{q^n[a - \varphi_J(0)] : \ J \in \mathcal{A}_n, \ \varphi_J[0,1] \cap (a,b) \neq \emptyset\}.$$

If $n = 0$, then we define $r_n(\Delta) = 1$. Else if $n \geq 1$, to define $r_n(\Delta)$, let $\widehat{\Delta}$ be the unique element of $\mathcal{F}_{n-1}$ containing $\Delta$. Order the intervals from the set

$$\left\{\Xi \in \mathcal{F}_n : \ \Xi \subseteq \widehat{\Delta}, \ \ell_n(\Xi) = \ell_n(\Delta), \ V_n(\Xi) = V_n(\Delta)\right\}$$

by their left end points. Then $r_n(\Delta)$ is defined to be the rank of $\Delta$ in this ordering.

We call the triple

$$\mathcal{C}_n(\Delta) \stackrel{\text{def}}{=} \langle \ell_n(\Delta),\ V_n(\Delta),\ r_n(\Delta) \rangle$$

the **characteristic vector** of $\Delta$, which encodes the length and neighborhood information of $\Delta$. Let

$$\Omega \stackrel{\text{def}}{=} \{\mathcal{C}_n(\Delta) : n \geq 0,\ \Delta \in \mathcal{F}_n\}$$

be the collection of all characteristic vectors. Given $\alpha = \langle c_1, c_2, c_3 \rangle \in \Omega$, we write

$$\ell(\alpha) \stackrel{\text{def}}{=} c_1, \quad V(\alpha) \stackrel{\text{def}}{=} c_2, \quad r(\alpha) \stackrel{\text{def}}{=} c_3.$$

If $\Phi$ satisfies the FTC, then we know from [21, Lemma 6.1] that $\Omega$ is a finite set.[1] In this case, let $\Omega^*$ denote the collection of all finite words over $\Omega$. We define a function $\zeta : \Omega \to \Omega^*$ as follows. Given $\alpha \in \Omega$, pick some $n$ and $\Delta \in \mathcal{F}_n$ such that $\alpha = \mathcal{C}_n(\Delta)$. Let $\Delta_1, \dots, \Delta_k$ (ordered by left end points increasingly) be all elements in $\mathcal{F}_{n+1}$ which are sub-intervals of $\Delta$, and write $\alpha_j := \mathcal{C}_{n+1}(\Delta_j)$. By [21, Lemma 6.1], the word $\alpha_1 \cdots \alpha_k$ depends on $\alpha$ only and is independent of the choice of $n$ and $\Delta$. Therefore, we can define

$$\zeta(\alpha) := \alpha_1 \cdots \alpha_k.$$

Associated with $\zeta$ is a matrix $M$ on $\Omega \times \Omega$ which is defined by

$$M_{\alpha,\beta} := \begin{cases} 1 & \text{if } \beta \text{ is a letter of } \zeta(\alpha) \\ 0 & \text{otherwise.} \end{cases}$$

We have $M \in \mathrm{Mat}(\{0,1\})$, where $\mathrm{Mat}(S)$ denotes the collection of all matrices $(a_{i,j})$ with $a_{i,j} \in S$.

After the above preparation, we are now ready to prove Theorem 1.1 (p.3). The proof is due to Feng (personal communication, 2018).

**Proof of Theorem 1.1**

Write $\Omega =: \{\alpha_1, \dots, \alpha_w\}$ and let $\mathbf{u}$ be the vector $(l_1, \dots, l_w) := (\ell(\alpha_1), \dots, \ell(\alpha_w))$. Given $\alpha \in \Omega$, pick some $n$ and $\Delta \in \mathcal{F}_n$ such that $\alpha = \mathcal{C}_n(\Delta)$. As

$$\rho^n \ell(\alpha) = \text{length of } \Delta = \sum_{\Xi \in \mathcal{F}_{n+1},\ \Xi \subseteq \Delta} (\text{length of } \Xi) = \sum_{\beta \in \Omega,\ M_{\alpha,\beta}=1} \rho^{n+1} \ell(\beta),$$

we have

$$q\mathbf{u} = q \begin{pmatrix} l_1 \\ \vdots \\ l_w \end{pmatrix} = M \begin{pmatrix} l_1 \\ \vdots \\ l_w \end{pmatrix} = M\mathbf{u}.$$

Note that $M \in \mathrm{Mat}(\mathbb{Z})$, $M \geq \mathbf{0}$, and $\mathbf{u} > \mathbf{0}$. Accordingly, we shall prove the result by making use of the theory of non-negative matrices (c.f. [51] for example).

[1] The converse also holds. One way to see this is that if $\Omega$ is a finite set, then the set $\{\ell_n(\Delta) : n \geq 0,\ \Delta \in \mathcal{F}_n\} = \{\ell(\alpha) : \alpha \in \Omega\}$ is bounded away from zero, so 0 is not an accumulation point of $Y$, and so $\Phi$ satisfies the FTC by [22, Theorem 1.11 and Lemma 2.1].

By rearranging $M$ into canonical form of irreducible components, we can write the above equation as

$$\text{(A.1)}\qquad \widetilde{M}\widetilde{\mathbf{u}} = \begin{pmatrix} T_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ P_{2,1} & T_2 & \mathbf{0} & \cdots & \mathbf{0} \\ P_{3,1} & P_{3,2} & T_3 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{R,1} & P_{R,2} & P_{R,3} & \cdots & T_R \end{pmatrix} \begin{pmatrix} \widetilde{\mathbf{u}}_1 \\ \vdots \\ \widetilde{\mathbf{u}}_R \end{pmatrix} = q \begin{pmatrix} \widetilde{\mathbf{u}}_1 \\ \vdots \\ \widetilde{\mathbf{u}}_R \end{pmatrix} = q\widetilde{\mathbf{u}},$$

where each $T_i$ is an irreducible matrix. From this equation we have $T_1\widetilde{\mathbf{u}}_1 = q\widetilde{\mathbf{u}}_1$. Since $T_1 \in \mathrm{Mat}(\mathbb{Z})$ and $\widetilde{\mathbf{u}}_1 \neq \mathbf{0}$, $q$ is an algebraic integer and all of its algebraic conjugates are eigenvalues of $T_1$. Moreover, as $T_1 \geq \mathbf{0}$ is irreducible, it follows that $|A| \leq q$ for all algebraic conjugate $A$ of $q$ (c.f. [51, Theorem 1.5 and 1.6]). It remains to show that $b_i \in \mathbb{Q}(q) = \mathbb{Q}[q]$ for all $i$.

As $T_1 \geq \mathbf{0}$ is irreducible, the solution space $\{\mathbf{x} : T_1\mathbf{x} = q\mathbf{x}\}$ has dimension one. Hence, noting that $T_1 \in \mathrm{Mat}(\mathbb{Z})$, from Gaussian elimination there exists $\mathbf{x}_1 \in \mathrm{Mat}(\mathbb{Q}(q))$ such that $T_1\mathbf{x}_1 = q\mathbf{x}_1$ and $\widetilde{\mathbf{u}}_1 = t\mathbf{x}_1$ for some $t \in \mathbb{R}$. Since $\widetilde{\mathbf{u}}_1 > \mathbf{0}$, we can take $\mathbf{x}_1 > \mathbf{0}$ and $t > 0$.

By [21, Lemma 6.4], $\Omega$ has exactly one essential class. Therefore, $P_{2,1} \geq \mathbf{0}, \neq \mathbf{0}$. Now (A.1) gives $(qI - T_2)\widetilde{\mathbf{u}}_2 = P_{2,1}\widetilde{\mathbf{u}}_1 = tP_{2,1}\mathbf{x}_1 =: \mathbf{y}_2$, where $\mathbf{y}_2 \geq \mathbf{0}, \neq \mathbf{0}$. As a result, $q$ is strictly greater than the Perron-Frobenius eigenvalue of $T_2$ (c.f. [51, Theorem 2.1]), whence $q$ is not an eigenvalue of $T_2$. This implies $\widetilde{\mathbf{u}}_2 = t(qI - T_2)^{-1}P_{2,1}\mathbf{x}_1 =: t\mathbf{x}_2$. We have $\mathbf{x}_2 \in \mathrm{Mat}(\mathbb{Q}(q))$. As $\widetilde{\mathbf{u}}_2 > \mathbf{0}$ and $t > 0$, the relation $\widetilde{\mathbf{u}}_2 = t\mathbf{x}_2$ gives $\mathbf{x}_2 > \mathbf{0}$.

Again, as $\Omega$ has exactly one essential class, we have $(P_{3,1}\ P_{3,2}) \geq \mathbf{0}, \neq \mathbf{0}$. (A.1) gives $(qI - T_3)\widetilde{\mathbf{u}}_3 = P_{3,1}\widetilde{\mathbf{u}}_1 + P_{3,2}\widetilde{\mathbf{u}}_2 = t(P_{3,1}\mathbf{x}_1 + P_{3,2}\mathbf{x}_2) =: \mathbf{y}_3$, where $\mathbf{y}_3 \geq \mathbf{0}, \neq \mathbf{0}$. The same argument as above gives $\widetilde{\mathbf{u}}_3 = t(qI - T_3)^{-1}(P_{3,1}\mathbf{x}_1 + P_{3,2}\mathbf{x}_2) =: t\mathbf{x}_3$, where $\mathbf{x}_3 \in \mathrm{Mat}(\mathbb{Q}(q))$ and $\mathbf{x}_3 > \mathbf{0}$. Continuing this way until the last row block of $\widetilde{M}$, we get

$$\text{(A.2)}\qquad \widetilde{\mathbf{u}}_i = t\mathbf{x}_i,\ \mathbf{x}_i \in \mathrm{Mat}(\mathbb{Q}(q)),\ \mathbf{x}_i > \mathbf{0}\quad (1 \leq i \leq R).$$

Observe that $\alpha := \mathcal{C}_0([0,1]) \in \Omega$ satisfies $\ell(\alpha) = 1$. Therefore, there exists $1 \leq i \leq R$ such that $\widetilde{\mathbf{u}}_i$ has an entry of value 1, whence (A.2) gives $t \in \mathbb{Q}(q)$ and $\widetilde{\mathbf{u}} \in \mathrm{Mat}(\mathbb{Q}(q))$. The same holds for $\mathbf{u} = (l_1, \ldots, l_w)$. Noting that the translation parameters $b_i$ of the IFS all belong to $\mathbb{Q}(l_1, \ldots, l_w, q)$, we finish the proof.

**Q.E.D.**

Finally, we give a proof of Theorem 1.9 (p.5), which was given by Feng [22] through an argument by Drobot [15].

**Proof of Theorem 1.9**

If $\Phi$ satisfies the FTC, then $Y$ has no accumulation point in $\mathbb{R}$, a fortiori $Y$ is not dense in $\mathbb{R}$.

Conversely, suppose $\Phi$ does not satisfy the FTC. Fix $r_0 \in \mathbb{R}$ and $\varepsilon_0 > 0$. We want to show that there exists $y \in Y$ such that $|y - r_0| < \varepsilon_0$. Since $Y = -Y$ and $0 \in Y$, we may assume $r_0 > 0$. We claim that it suffices to show the following result:

(♘) Given any $r > 0$, and any positive integer $M > 0$, there exists $y := \sum_{i=a(y)}^{b(y)} q^i s_i \in Y$, where $s_i = s_i(y) \in (\mathsf{D}(q) - \mathsf{D}(q))$ and $a(y) > M$, such that $0 < r - y \leq r(1 - 1/(2q))$.

For, suppose it is true. We obtain $y_1, \dots, y_N \in Y$ by applying it $N$ times, in the way that letting $r_\ell := r_0 - \sum_{i=1}^{\ell} y_i$, we go through the following steps successively:

$$\begin{array}{lll}
\text{taking } (r, M) := (r_0, 1) & \longrightarrow & \begin{cases} y_1 = \sum_{i=a(y_1)}^{b(y_1)} q^i s_i(y_1), \ a(y_1) > 1 \\ 0 < r_0 - y_1 \le r_0(1 - 1/(2q)), \end{cases} \\
\text{taking } (r, M) := (r_1, b(y_1)) & \longrightarrow & \begin{cases} y_2 = \sum_{i=a(y_2)}^{b(y_2)} q^i s_i(y_2), \ a(y_2) > b(y_1) \\ 0 < r_1 - y_2 \le r_1(1 - 1/(2q)), \end{cases} \\
\qquad\vdots & & \qquad\qquad\vdots \\
\text{taking } (r, M) := (r_{N-1}, b(y_{N-1})) & \longrightarrow & \begin{cases} y_N = \sum_{i=a(y_N)}^{b(y_N)} q^i s_i(y_N), \ a(y_N) > b(y_{N-1}) \\ 0 < r_{N-1} - y_N \le r_{N-1}(1 - 1/(2q)). \end{cases}
\end{array}$$

Since $a(y_{i+1}) > b(y_i)$ for all $i$, we have $y_* := y_1 + \cdots + y_N \in Y$. Also, we have $r_0 - y_* = r_{N-1} - y_N$, whence

$$\begin{aligned}
0 < r_0 - y_* &\le r_{N-1}(1 - 1/(2q)) = (r_{N-2} - y_{N-1})(1 - 1/(2q)) \\
&\le r_{N-2}(1 - 1/(2q))^2 = (r_{N-3} - y_{N-2})(1 - 1/(2q))^2 \\
&\le r_{N-3}(1 - 1/(2q))^3 = \cdots \\
&\le r_0(1 - 1/(2q))^N.
\end{aligned}$$

As the last expression is less than $\varepsilon_0$ when $N$ is large, we have verified our claim.

It remains to show (♘). Fix $r > 0$ and $M \in \mathbb{N}$. Since $\Phi$ does not satisfy the FTC, by [22, Theorem 1.11 and Lemma 2.1], 0 is an accumulation point of $Y$. Our argument now follows from [15]. Fix a large positive integer $u_0 \in \mathbb{N}$ such that

$$\text{(A.3)} \qquad \frac{1}{q^{u_0}} < r \qquad \text{and} \qquad r + \frac{1}{q^{u_0}} - \frac{1}{q}\left(r - \frac{1}{q^{u_0}}\right) \le r\left(1 - \frac{1}{2q}\right).$$

Given $n \in \mathbb{N}$, since $Y = -Y$ and 0 is an accumulation point of $Y$, there exists $z \in Y$ such that

$$0 < z < \frac{1}{q^{u_0+n}}.$$

Accordingly, there exists $v \ge n$ such that

$$\frac{1}{q^{u_0+v+1}} \le z < \frac{1}{q^{u_0+v}}, \qquad \text{therefore} \qquad \frac{1}{q^{u_0+1}} \le z \cdot q^v < \frac{1}{q^{u_0}}.$$

Observe that $q^v Y \subseteq Y$. Hence, we have shown that: given any $n \in \mathbb{N}$, there exists $\gamma(n) := \sum_{i=h(n)}^{k(n)} q^i s_i \in Y$, where $s_i \in (\mathsf{D}(q) - \mathsf{D}(q))$ and $h(n) > n$, such that $q^{-u_0-1} \le \gamma(n) < q^{-u_0}$.

By (A.3), there exists $N_0 \ge 1$ such that

$$\text{(A.4)} \qquad \frac{N_0}{q^{u_0}} \le r \le \frac{N_0 + 1}{q^{u_0}}.$$

Define $g_1 := \gamma(M)$, $g_2 := \gamma(k(M))$, $g_3 := \gamma(k(k(M)))$ and so on up to $g_{N_0}$. Note that $y := \sum_{j=1}^{N_0} g_j \in Y$, and we can write $y = \sum_{i=a(y)}^{b(y)} q^i s_i$ with $s_i \in (\mathsf{D}(q) - \mathsf{D}(q))$ and $a(y) > M$. Moreover, by the property of $\gamma(n)$, we have

$$y \in \left[\frac{N_0}{q^{u_0+1}}, \frac{N_0}{q^{u_0}}\right).$$

Together with (A.4), we have

$$\begin{aligned} 0 < r - y &\le \left(\frac{N_0 + 1}{q^{u_0}}\right) - \frac{N_0}{q^{u_0+1}} \\ &\le \left(r + \frac{1}{q^{u_0}}\right) - \frac{N_0}{q^{u_0+1}} \\ &\le \left(r + \frac{1}{q^{u_0}}\right) - \frac{1}{q}\left(r - \frac{1}{q^{u_0}}\right) \\ &\le r\left(1 - \frac{1}{2q}\right) \qquad \text{by (A.3).} \end{aligned}$$

This shows (♘) and completes the proof.

**Q.E.D.**

# Appendix B

# Symbolic computation of systems of equations

During the research, we find examples and make conjectures by solving various systems of equations. Here we demonstrate how we use GNU Octave 5.2.0 for the symbolic computation of (5.6) (p.74). The purpose is to find a 5-tuple IFS such that the associated $q$ is not a PV number. Below is the code we use. We only check the range $1 \leq h, k \leq 10$. Inside the code, we ensure the following conditions hold:

1. $q > 3$. It is because if $q \leq 3$, then $[2; 0] = (q - 1)/2 \leq 1$, whence the IFS can be reduced from a 5-tuple to a 4-tuple by dropping $\partial_1$.
2. $q$ has a non-real algebraic conjugate $A$ with $|A| \geq 1$.
3. We have $|A| < |1 + \sigma_A(a)|$.
4. We have
$$\left|\frac{\sigma_A(a) + 1}{A}\right| \geq |\sigma_A(y) - 1|$$
for $y = [1; 0]$ and $[2; 1]$.

```
pkg load symbolic
syms t A

myPrecision = 0.0001;

for h = 1:10
    eq1 = t - (A^h - 1)/(A^h + 1) == 0;
    tExpress = solve(eq1, t);

    for k = 1:10
        eq2 = A-1 - 2*t - 2*( A^k - 1 - t )/A^k == 0;

        myAns = factor(subs(eq2, t, tExpress));

        myPolyCoeffs = sym2poly(numden(lhs(myAns)));

        myRoots = roots(myPolyCoeffs);

```

```
        myRootAbs = abs(myRoots);

        q=1;
        algeConj = 1;
        isSetAlgeConj = false;
        gtOneCount = 0;
        for myAbsIndex = 1 : length(myRootAbs)

            if(myRootAbs(myAbsIndex) > 1 - myPrecision)
                gtOneCount = gtOneCount+1;

                if(abs(imag(myRoots(myAbsIndex))) < myPrecision && ...
                    real(myRoots(myAbsIndex)) > 3 )
                    q = myRoots(myAbsIndex);
                end

                if(~isSetAlgeConj && abs(imag(myRoots(myAbsIndex))) ...
                    > myPrecision)
                    algeConj=myRoots(myAbsIndex);
                    isSetAlgeConj = true;
                end

            end

        end

        if(q>1 && gtOneCount > 2 && isSetAlgeConj)

            funTau = function_handle(tExpress);
            tau = funTau(algeConj);
            a = funTau(q);

            if( true
                && abs(algeConj) < abs(1+tau) + myPrecision  %|A| < ...
                    |1+t|
                && abs((1+tau)/algeConj) > abs(tau-1) - myPrecision ...
                    % |(1+t)/A| >= |t-1|
                && abs((1+tau)/algeConj) > abs((algeConj-1)/2-tau-1) ...
                    - myPrecision % |(1+t)/A| >= |(A-1)/2-t-1|
                )
                disp(" #### My Choice: #### ");
                h
                k
                myAns
                myPolyCoeffs
                myRoots
                myRootAbs
                q
                algeConj
                a
                tau
                disp(" ")
            end
        end
    end
end
```

Below is an excerpt of the output. It shows the data for all $h \leq 6$.

```
 #### My Choice: ####
h =  2
k =  5
myAns = (sym)

            / 5        4        3        2           \
  (A - 1)*\A   - 4*A   - 3*A   - 4*A   - 4*A - 4/
  ------------------------------------------------------ = 0
                          3 / 2     \
                         A *\A  + 1/

myPolyCoeffs =

    1   -5    1   -1    0    0    4

myRoots =

    4.83442 + 0.00000i
    1.00000 + 0.00000i
    0.28501 + 0.97679i
    0.28501 - 0.97679i
   -0.70222 + 0.55321i
   -0.70222 - 0.55321i

myRootAbs =

    4.83442
    1.00000
    1.01752
    1.01752
    0.89395
    0.89395

q =  4.8344
algeConj =  0.28501 + 0.97679i
a =  0.91794
tau =  0.22056 + 3.41411i

 #### My Choice: ####
h =  2
k =  6
myAns = (sym)

            / 6        5        4        3        2           \
  (A - 1)*\A   - 4*A   - 3*A   - 4*A   - 4*A   - 4*A - 4/
  ---------------------------------------------------------------- = 0
                              4 / 2     \
                             A *\A  + 1/

myPolyCoeffs =

    1   -5    1   -1    0    0    0    4

myRoots =

    4.83566 + 0.00000i
    1.00000 + 0.00000i
    0.47421 + 0.89959i
```

```
   0.47421 - 0.89959i
  -0.44923 + 0.83748i
  -0.44923 - 0.83748i
  -0.88563 + 0.00000i

myRootAbs =

   4.83566
   1.00000
   1.01693
   1.01693
   0.95035
   0.95035
   0.88563

q =  4.8357
algeConj =  0.47421 + 0.89959i
a =  0.91798
tau =  0.077114 + 1.894563i

 #### My Choice: ####
h =  2
k =  7
myAns = (sym)

               / 7        6        5        4        3        2          \
  (A - 1)*\A  - 4*A  - 3*A  - 4*A  - 4*A  - 4*A  - 4*A - 4/
  ---------------------------------------------------------------- = 0
                                  5  / 2     \
                                 A *\A  + 1/

myPolyCoeffs =

   1  -5   1  -1   0   0   0   0   4

myRoots =

   4.83591 + 0.00000i
   1.00000 + 0.00000i
   0.60139 + 0.81532i
   0.60139 - 0.81532i
  -0.20496 + 0.96185i
  -0.20496 - 0.96185i
  -0.81438 + 0.41231i
  -0.81438 - 0.41231i

myRootAbs =

   4.83591
   1.00000
   1.01312
   1.01312
   0.98345
   0.98345
   0.91281
   0.91281

q =  4.8359
```

```
algeConj =  0.60139 + 0.81532i
a =  0.91799
tau =  0.036971 + 1.355079i

 #### My Choice: ####
h =  2
k =  8
myAns = (sym)

             / 6        5        4              \
  (A - 1)*\A   - 4*A   - 4*A   - 4*A - 4/
  ----------------------------------------------- = 0
                          6
                         A

myPolyCoeffs =

    1  -5    0    4    0   -4    0    4

myRoots =

    4.83596 + 0.00000i
    1.00000 + 0.00000i
    0.68932 + 0.73775i
    0.68932 - 0.73775i
   -0.65025 + 0.68175i
   -0.65025 - 0.68175i
   -0.91411 + 0.00000i

myRootAbs =

    4.83596
    1.00000
    1.00967
    1.00967
    0.94213
    0.94213
    0.91411

q =  4.8360
algeConj =  0.68932 + 0.73775i
a =  0.91799
tau =  0.020649 + 1.070040i

 #### My Choice: ####
h =  2
k =  9
myAns = (sym)

             / 9        8        7        6        5        4        3        2 ...
                                               \
  (A - 1)*\A   - 4*A   - 3*A   - 4*A   - 4*A   - 4*A   - 4*A   - 4*A   - 4*A ...
     - 4/
  ------------------------------------------------------------------------------------------------ ...
     = 0
                                            7  / 2      \
                                           A  *\A   + 1/

```

```
myPolyCoeffs =

   1  -5   1  -1   0   0   0   0   0   0   4

myRoots =

   4.83597 + 0.00000i
   1.00000 + 0.00000i
   0.75198 + 0.66993i
   0.75198 - 0.66993i
   0.16509 + 0.99346i
   0.16509 - 0.99346i
  -0.46622 + 0.84562i
  -0.46622 - 0.84562i
  -0.86884 + 0.32779i
  -0.86884 - 0.32779i

myRootAbs =

   4.83597
   1.00000
   1.00712
   1.00712
   1.00708
   1.00708
   0.96563
   0.96563
   0.92861
   0.92861

q =  4.8360
algeConj =  0.75198 + 0.66993i
a =  0.91799
tau =  0.012721 + 0.890809i

 #### My Choice: ####
h =  2
k =  10
myAns = (sym)

          / 10      9      8      7      6      5      4      3      2
  (A - 1)*\A    - 4*A  - 3*A  - 4*A  - 4*A  - 4*A  - 4*A  - 4*A  - ...
      4*A  - 4*A - 4
  ------------------------------------------------------------------------
                                       8 / 2    \
                                      A *\A  + 1/

  \
  /
  - = 0



myPolyCoeffs =

   1  -5   1  -1   0   0   0   0   0   0   0   4

myRoots =
```

```

   4.83598 + 0.00000i
   1.00000 + 0.00000i
   0.79790 + 0.61156i
   0.79790 - 0.61156i
   0.29680 + 0.96472i
   0.29680 - 0.96472i
  -0.29136 + 0.93814i
  -0.29136 - 0.93814i
  -0.75572 + 0.56820i
  -0.75572 - 0.56820i
  -0.93121 + 0.00000i

myRootAbs =

   4.83598
   1.00000
   1.00531
   1.00531
   1.00934
   1.00934
   0.98235
   0.98235
   0.94550
   0.94550
   0.93121

q = 4.8360
algeConj = 0.79790 + 0.61156i
a = 0.91799
tau = 0.0084051 + 0.7664279i

 #### My Choice: ####
h = 3
k = 7
myAns = (sym)

            / 7        6        5        4        3        2            \
  (A - 1)*\A  - 4*A  - 4*A  - 3*A  - 4*A  - 4*A  - 4*A - 4/
  ------------------------------------------------------------------- = 0
                        4             / 2          \
                       A *(A + 1)*\A  - A + 1/

myPolyCoeffs =

   1  -5   0   1  -1   0   0   0   4

myRoots =

   4.96758 + 0.00000i
   1.00000 + 0.00000i
   0.62233 + 0.79529i
   0.62233 - 0.79529i
  -0.89303 + 0.36456i
  -0.89303 - 0.36456i
  -0.21309 + 0.89624i
  -0.21309 - 0.89624i

```

```
myRootAbs =

   4.96758
   1.00000
   1.00984
   1.00984
   0.96458
   0.96458
   0.92123
   0.92123

q =  4.9676
algeConj =  0.62233 + 0.79529i
a =  0.98382
tau =  0.33454 + 4.65507i

 #### My Choice: ####
h =  3
k =  8
myAns = (sym)

              / 7      6     5      4      2    \
  (A - 1)*\A  - 5*A  + A  - 4*A  - 4*A  - 4/
  ---------------------------------------------- = 0
                     5 / 2         \
                    A *\A  - A + 1/

myPolyCoeffs =

   1  -6   6  -5   4  -4   4  -4   4

myRoots =

   4.96762 + 0.00000i
   1.00000 + 0.00000i
   0.70564 + 0.72551i
   0.70564 - 0.72551i
   0.00453 + 0.94872i
   0.00453 - 0.94872i
  -0.69398 + 0.62591i
  -0.69398 - 0.62591i

myRootAbs =

   4.96762
   1.00000
   1.01207
   1.01207
   0.94873
   0.94873
   0.93455
   0.93455

q =  4.9676
algeConj =  0.70564 + 0.72551i
a =  0.98382
tau =  0.13600 + 2.55767i

```

```
 #### My Choice: ####
h =  3
k =  9
myAns = (sym)

             /  9         8         7         6         5         4         3         2 ...
                               \
  (A - 1)*\A   - 4*A   - 4*A   - 3*A   - 4*A   - 4*A   - 4*A   - 4*A   - 4*A ...
      - 4/
  -------------------------------------------------------------------------------------- ...
      = 0
                                      6           /  2        \
                                     A  *(A + 1)*\A   - A + 1/

myPolyCoeffs =

    1   -5    0    1   -1    0    0    0    0    0    4

myRoots =

    4.96763 + 0.00000i
    1.00000 + 0.00000i
    0.76453 + 0.66223i
    0.76453 - 0.66223i
    0.17572 + 0.95398i
    0.17572 - 0.95398i
   -0.93453 + 0.29683i
   -0.93453 - 0.29683i
   -0.48954 + 0.79394i
   -0.48954 - 0.79394i

myRootAbs =

    4.96763
    1.00000
    1.01147
    1.01147
    0.97003
    0.97003
    0.98054
    0.98054
    0.93273
    0.93273

q =  4.9676
algeConj =  0.76453 + 0.66223i
a =  0.98382
tau =  0.074306 + 1.827876i

 #### My Choice: ####
h =  3
k =  10
myAns = (sym)

             /  9         8      7         6         4         2     \
  (A - 1)*\A   - 5*A   + A   - 4*A   - 4*A   - 4*A   - 4/
  ------------------------------------------------------------ = 0
                            7  /  2          \
```

```
                        A *\A   - A + 1/

myPolyCoeffs =

    1   -6    6   -5    4   -4    4   -4    4   -4    4

myRoots =

    4.96763 + 0.00000i
    1.00000 + 0.00000i
    0.80757 + 0.60657i
    0.80757 - 0.60657i
    0.30994 + 0.93464i
    0.30994 - 0.93464i
   -0.79886 + 0.52788i
   -0.79886 - 0.52788i
   -0.30247 + 0.89244i
   -0.30247 - 0.89244i

myRootAbs =

    4.96763
    1.00000
    1.01000
    1.01000
    0.98469
    0.98469
    0.95751
    0.95751
    0.94230
    0.94230

q =   4.9676
algeConj =   0.80757 + 0.60657i
a =   0.98382
tau =   0.046163 + 1.446707i

 #### My Choice: ####
h =  4
k =  3
myAns = (sym)

             /  4         3         2              \
   (A - 1)*\A   - 4*A   - 4*A   - 4*A + 1/
   --------------------------------------------- = 0
                         4
                        A   + 1

myPolyCoeffs =

    1   -5    0    0    5   -1

myRoots =

    4.96069 + 0.00000i
   -0.58114 + 0.81380i
   -0.58114 - 0.81380i
    1.00000 + 0.00000i
```

```
    0.20158 + 0.00000i

myRootAbs =

    4.96069
    1.00000
    1.00000
    1.00000
    0.20158

q =  4.9607
algeConj = -0.58114 + 0.81380i
a =  0.99670
tau = -9.7067e-15 + 2.9143e+00i

 #### My Choice: ####
h =  4
k =  9
myAns = (sym)

             /  9        8        7        6        5        4        3        2 ...
                                    \
  (A - 1)*\A  - 4*A  - 4*A  - 4*A  - 3*A  - 4*A  - 4*A  - 4*A  - 4*A ...
      - 4/
  ------------------------------------------------------------------------------ ...
      = 0
                                                        5 / 4     \
                                                       A *\A  + 1/

myPolyCoeffs =

    1  -5   0   0   1  -1   0   0   0   0   4

myRoots =

    4.99358 + 0.00000i
    1.00000 + 0.00000i
    0.76784 + 0.64903i
    0.76784 - 0.64903i
    0.14396 + 0.93614i
    0.14396 - 0.93614i
   -0.53277 + 0.84106i
   -0.53277 - 0.84106i
   -0.87582 + 0.35233i
   -0.87582 - 0.35233i

myRootAbs =

    4.99358
    1.00000
    1.00539
    1.00539
    0.94714
    0.94714
    0.99560
    0.99560
    0.94403
    0.94403
```

```

q =   4.9936
algeConj =   0.76784 + 0.64903i
a =   0.99679
tau =   0.38637 + 5.89609i

 #### My Choice: ####
h =   4
k =   10
myAns = (sym)

               / 10         9         8         7         6         5         4         3         2
   (A - 1)*\A    - 4*A   - 4*A   - 4*A   - 3*A   - 4*A   - 4*A   - 4*A   - ...
       4*A   - 4*A - 4
   ------------------------------------------------------------------------------------------------
                                                                      6  /  4      \
                                                                     A  *\A   + 1/

   \
   /
   - = 0



myPolyCoeffs =

    1   -5    0    0    1   -1    0    0    0    0    0    4

myRoots =

    4.99358 + 0.00000i
    1.00000 + 0.00000i
    0.81142 + 0.59705i
    0.81142 - 0.59705i
    0.28909 + 0.91521i
    0.28909 - 0.91521i
   -0.34784 + 0.91067i
   -0.34784 - 0.91067i
   -0.78596 + 0.59573i
   -0.78596 - 0.59573i
   -0.92700 + 0.00000i

myRootAbs =

    4.99358
    1.00000
    1.00741
    1.00741
    0.95978
    0.95978
    0.97484
    0.97484
    0.98622
    0.98622
    0.92700

q =   4.9936
algeConj =   0.81142 + 0.59705i
```

```
a =  0.99679
tau =  0.16641 + 3.20103i

 #### My Choice: ####
h =  5
k =  4
myAns = (sym)

                 / 4        3      2            \
   (A - 1)*\A  - 5*A  + A  - 5*A + 1/
  ------------------------------------------ = 0
                  4      3      2
                 A  - A  + A  - A + 1

myPolyCoeffs =

    1  -6   6  -6   6  -1

myRoots =

    4.99227 + 0.00000i
   -0.09629 + 0.99535i
   -0.09629 - 0.99535i
    1.00000 + 0.00000i
    0.20031 + 0.00000i

myRootAbs =

    4.99227
    1.00000
    1.00000
    1.00000
    0.20031

q =  4.9923
algeConj = -0.096291 + 0.995353i
a =  0.99936
tau = -2.2011e-15 + 1.6521e+00i

 #### My Choice: ####
h =  5
k =  7
myAns = (sym)

                 / 7        6        5        4        3        2          \
   (A - 1)*\A  - 4*A  - 4*A  - 4*A  - 4*A  - 3*A  - 4*A - 4/
  ---------------------------------------------------------------------- = 0
                        2           / 4      3      2         \
                       A *(A + 1)*\A  - A  + A  - A + 1/

myPolyCoeffs =

    1  -5   0   0   0   1  -1   0   4

myRoots =

    4.99867 + 0.00000i
    1.00000 + 0.00000i
```

```
    0.59884 + 0.75480i
    0.59884 - 0.75480i
   -0.24259 + 0.97690i
   -0.24259 - 0.97690i
   -0.85558 + 0.34459i
   -0.85558 - 0.34459i

myRootAbs =

    4.99867
    1.00000
    0.96350
    0.96350
    1.00657
    1.00657
    0.92237
    0.92237

q =  4.9987
algeConj = -0.24259 + 0.97690i
a =  0.99936
tau =  0.52441 + 5.54584i

 #### My Choice: ####
h =  6
k =  5
myAns = (sym)

             / 6        5        4        3        2           \
   (A - 1)*\A   - 4*A   - 4*A   - 4*A   - 4*A   - 4*A + 1/
   -------------------------------------------------------------- = 0
                     / 2      \ / 4      2      \
                     \A   + 1/*\A   - A   + 1/

myPolyCoeffs =

    1   -5    0    0    0    0    5   -1

myRoots =

    4.99846 + 0.00000i
   -0.83080 + 0.55657i
   -0.83080 - 0.55657i
    0.23154 + 0.97283i
    0.23154 - 0.97283i
    1.00000 + 0.00000i
    0.20006 + 0.00000i

myRootAbs =

    4.99846
    1.00000
    1.00000
    1.00000
    1.00000
    1.00000
    0.20006
```

```
q =   4.9985
algeConj = -0.83080 + 0.55657i
a =   0.99987
tau = -1.5065e-14 + 4.9341e+00i

 #### My Choice: ####
h =   6
k =   9
myAns = (sym)

              / 9         8         7         6         5         4         3         2 ...
                                 \
  (A - 1)*\A   - 4*A   - 4*A   - 4*A   - 4*A   - 4*A   - 3*A   - 4*A   - 4*A ...
      - 4/
  ------------------------------------------------------------------------------------- ...
      = 0
                                    3  /  2       \  /  4       2       \
                                   A  *\A   + 1/*\A   - A   + 1/

myPolyCoeffs =

    1   -5    0    0    0    0    1   -1    0    0    4

myRoots =

    4.99974 + 0.00000i
    1.00000 + 0.00000i
    0.75140 + 0.63173i
    0.75140 - 0.63173i
    0.14495 + 0.99711i
    0.14495 - 0.99711i
   -0.90460 + 0.36583i
   -0.90460 - 0.36583i
   -0.49162 + 0.78560i
   -0.49162 - 0.78560i

myRootAbs =

    4.99974
    1.00000
    0.98167
    0.98167
    1.00759
    1.00759
    0.97577
    0.97577
    0.92674
    0.92674

q =   4.9997
algeConj =   0.14495 + 0.99711i
a =   0.99987
tau =   0.12851 + 2.15657i

......

>>
```

# Appendix C

# Numerical computation of characteristic vectors

We use Scilab 6.1.0 for the numerical computation of characteristic vectors. Below is the code we use. Inside the code, we make an assumption that the length of the second components of the characteristic vectors are less than 10. The constant `typeVectorAllowedLength` corresponds to this upper bound. The meanings of the parameters of this program are as follows.

1. `contractInv`: the value of $q$.
2. `myDigitSet`: the value of $\{\partial_0, \dots, \partial_m\}$.
3. `testLv`: the number of iterations of the IFS we would like to examine.
4. `outputType`: "0" means listing all characteristic vectors given by the 1-st, 2-nd, . . . , `testLv`-th basic net intervals. "1" means just listing the characteristic vectors given by the `testLv`-th basic net intervals.

```
function [x]=listCharVect(contractInv, myDigitSet, testLv, outputType)

//outputType:
//0 - default
//1 - last testLv Only

myPrecision = 0.000001;
typeVectorAllowedLength = 10;

// a fundamental interval is an n-th basic net interval times q^n ...
    for some n
fundamentalIntervals = [];

// the characteristic vectors will be stored in the below array
typeCoding = [];
typeCodingIndex = 1;

currentLv = 0;
prevLvCombine = [0];
thisLvCombine = [];

```

```
// check correct normalization
if ( abs( myDigitSet(length(myDigitSet)) - (contractInv -1) ) > ...
    myPrecision )
    disp("Wrong normalization");
    return;
end

// check if the intervals overlap
for i=2:length(myDigitSet)
    if myDigitSet(i)-myDigitSet(i-1)>= 1 + myPrecision
        disp("Gap too large:" + string(myDigitSet(i)) +"-"+ ...
            string(myDigitSet(i-1)));
        return;
    end
end

while currentLv < testLv
    currentLv = currentLv+1;

    disp("#################### Level " + string(currentLv) + " ...
        ########################");

    thisLvTailArray = contractInv * prevLvCombine;

    thisLvCombine = [];
    thisLvCombineIndex = 1;

    if currentLv==1
        thisLvCombine = myDigitSet;
    else
        for digitIndex = 1:length(myDigitSet)
            for thisLvTailIndex = 1:length(thisLvTailArray)

                thisLvCombine(thisLvCombineIndex) = ...
                    myDigitSet(digitIndex) + ...
                    thisLvTailArray(thisLvTailIndex);

                thisLvCombineIndex = thisLvCombineIndex+1;
            end
        end
    end

    thisLvCombine = uniqueAscArray(gsort(thisLvCombine, 'g', 'i'), ...
        myPrecision); // 'i' means ascending
    rightEndPoints = thisLvCombine + 1;
    allEndPoints = uniqueAscArray(gsort([thisLvCombine ...
        rightEndPoints], 'g', 'i'), myPrecision);

    prevLvCombine = thisLvCombine;

    if(outputType==1 && currentLv < testLv )
        continue;
    end

    oriIntervalFirstReachIndex = 1;
    for endPointIndex = 1:length(allEndPoints)-1

        fundaLeftEnd = allEndPoints(endPointIndex);
```

```
        fundaRightEnd = allEndPoints(endPointIndex+1);

        fundamentalIntervals(currentLv, endPointIndex, 1) = ...
            fundaLeftEnd;
        fundamentalIntervals(currentLv, endPointIndex, 2) = ...
            fundaRightEnd;

        type2ndCompForThisFunda = [];

        firstReach = 0;
        for oriIntervalLeftEndIndex = oriIntervalFirstReachIndex: ...
            length(thisLvCombine)

            // an original interval is of the form ...
                [d0+q*d1+...+q^n*dn, d0+q*d1+...+q^n*dn+1]
            oriIntervalLeftEnd = ...
                thisLvCombine(oriIntervalLeftEndIndex);
            oriIntervalRightEnd = oriIntervalLeftEnd+1;

            if(oriIntervalLeftEnd>fundaRightEnd)
                break;
            end

            // if the left end point of the funda interval equals ...
                the left end point of an original interval
            if ( abs( oriIntervalLeftEnd - fundaLeftEnd ...
                )<myPrecision )

                type2ndCompForThisFunda = [type2ndCompForThisFunda 0];

                if(firstReach == 0)
                    firstReach = 1;
                end

            // if an original interval contains the funda interval
            elseif ( ( oriIntervalLeftEnd < fundaLeftEnd + ...
                myPrecision ) && ( fundaRightEnd < ...
                oriIntervalRightEnd + myPrecision ) && ( abs( ...
                oriIntervalRightEnd - fundaLeftEnd )> myPrecision ) )

                type2ndCompForThisFunda = [type2ndCompForThisFunda ...
                    fundaLeftEnd-oriIntervalLeftEnd];

                if(firstReach == 0)
                    firstReach = 1;
                end

            end

            if(firstReach == 1)
                oriIntervalFirstReachIndex = oriIntervalLeftEndIndex;
                firstReach = 2;
            end

        end

        // store the characteristic vector
        typeCoding(typeCodingIndex,:) = ...
```

```
            [(fundaRightEnd-fundaLeftEnd) type2ndCompForThisFunda ...
            -1*ones(1, typeVectorAllowedLength - ...
            length(type2ndCompForThisFunda))];
        typeCodingIndex = typeCodingIndex+1;

    end
end

x = uniqueType(typeCoding, myPrecision);
disp("Size: " + string(size(x,1)));

endfunction

function [rtn]= uniqueAscArray(input, precision)
    rtn = [0];
    for inputIndex = 1:length(input)
        if ( abs( input(inputIndex) - rtn(length(rtn)) ) > precision )
            rtn = [rtn input(inputIndex)];
        end
    end
endfunction

function rtn= myGsortFun(input)
    rtn = int(100000*input);
endfunction

function [rtn]= uniqueType(input, precision)

    rtn = [];
    rtnNo = 0;

    for inputIndex = 1:length(input(:,1))

        isConsidered = 0;
        for tempRtnIndex = 1:length(rtn(:,1))
            if ( abs( input(inputIndex,1) - rtn(tempRtnIndex,1) ) < ...
                precision )

                allEqual = 1;
                for remainingRowIndex = 2:length(input(inputIndex,:))
                    if ( abs( input(inputIndex,remainingRowIndex) - ...
                        rtn(tempRtnIndex,remainingRowIndex) ) > ...
                        precision )
                        allEqual = 0;
                        break;
                    end
                end

                if (allEqual ==1)
                    isConsidered = 1;
                end
            end
        end

        if (isConsidered == 0)
            rtnNo = rtnNo +1;
            rtn(rtnNo, :) = input(inputIndex,:);
        end
```

```
    end

    rtn = gsort(rtn, 'lr', 'i', list(myGsortFun));

endfunction
```

Sample output for Chapter 5 Example V is as follows. It demonstrates numerically that, ignoring the third components of the characteristic vectors, the 1st iteration of the IFS gives rise to seven characteristic vectors, the 2nd iteration gives three more, so does the 3rd, and there is no new characteristic vector starting from the 4th iteration.

```

--> q = roots([1 -4 -4 -4 1])(1)
 q  =

   4.9606929

--> a = (q^4-1)/(q^4+1)
 a  =

   0.9967028

--> [x] = listCharVect(q, [0 a (q-1)/2 q-1-a q-1], 1, 0)

  "##################### Level 1 ########################"

  "Size: 7"
 x  =

   0.0032972   0.9967028   0.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.0163564   0.9836436   0.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.9672873   0.0163564  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.9803465   0.0032972  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.9803465   0.0163564  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.9967028   0.         -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.9967028   0.0032972  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.

--> [x] = listCharVect(q, [0 a (q-1)/2 q-1-a q-1], 2, 0)

  "##################### Level 1 ########################"

  "##################### Level 2 ########################"

  "Size: 10"
 x  =

   0.0032972   0.9967028   0.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.0163564   0.9836436   0.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.0811388   0.9188612   0.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.915564    0.0032972  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.915564    0.0811388  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.9672873   0.0163564  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.9803465   0.0032972  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.9803465   0.0163564  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.9967028   0.         -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.9967028   0.0032972  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
```

```
--> [x] = listCharVect(q, [0 a (q-1)/2 q-1-a q-1], 3, 0)

 "#################### Level 1 #######################"

 "#################### Level 2 #######################"

 "#################### Level 3 #######################"

 "Size: 13"
 x =

   0.0032972   0.9967028   0.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.0163564   0.9836436   0.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.0811388   0.9188612   0.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.4025048   0.5974952   0.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.594198    0.0032972  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.594198    0.4025048  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.915564    0.0032972  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.915564    0.0811388  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.9672873   0.0163564  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.9803465   0.0032972  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.9803465   0.0163564  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.9967028   0.         -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.9967028   0.0032972  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.

--> [x] = listCharVect(q, [0 a (q-1)/2 q-1-a q-1], 4, 0)

 "#################### Level 1 #######################"

 "#################### Level 2 #######################"

 "#################### Level 3 #######################"

 "#################### Level 4 #######################"

 "Size: 13"
 x =

   0.0032972   0.9967028   0.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.0163564   0.9836436   0.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.0811388   0.9188612   0.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.4025048   0.5974952   0.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.594198    0.0032972  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.594198    0.4025048  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.915564    0.0032972  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.915564    0.0811388  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.9672873   0.0163564  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.9803465   0.0032972  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.9803465   0.0163564  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.9967028   0.         -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
   0.9967028   0.0032972  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.  -1.
```

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