# NUMERATION SYSTEMS WITH

FINITE TYPE CONDITION

(Part II: Demonstration)

Demonstration

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FTC and  $y = \sum_{i=1}^{\infty} \frac{t_i}{q^i}$ , where  $t_i \in (\mathsf{D} - \mathsf{D})$ . We have

(a) 
$$\{yq^n - t_1q^{n-1} - \cdots - t_n : n \ge 0\}$$
 is a finite set.  
(b) If  $|A| > 1$ , then  $\sigma_A(y) = \sum_{i=1}^{\infty} \frac{\sigma_A(t_i)}{A^i}$ .

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## Proof of item(a)

FTC and  $y = \sum_{i=1}^{\infty} \frac{t_i}{\sigma^i}$ , where  $t_i \in (D-D)$ . We have (a)  $\{yq^n - t_1q^{n-1} - \cdots - t_n : n \ge 0\}$  is a finite set.

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## $\left[ \frac{t_1}{q} + \dots + \frac{t_n}{q^n} + \sum_{n+1}^{\infty} \frac{t_i}{q^i} = y \right]$

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(b) If 
$$|A| > 1$$
, then  $\sigma_A(y) = \sum_{i=1}^{\infty} \frac{\sigma_A(t_i)}{A^i}$ .

#### **Proof of item(a)**

$$\frac{1}{\left[\frac{t_1}{q} + \dots + \frac{t_n}{q^n} + \sum_{n+1}^{\infty} \frac{t_i}{q^i} = y\right]} \Rightarrow \left[q^n \sum_{n+1}^{\infty} \frac{t_i}{q^i} = \left(yq^n - t_1q^{n-1} - \dots - t_n\right)\right].$$

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L.H.S.:

 $\left[\frac{t_1}{q} + \dots + \frac{t_n}{q^n} + \sum_{n+1}^{\infty} \frac{t_i}{q^i} = y\right] \Rightarrow \left[q^n \sum_{n+1}^{\infty} \frac{t_i}{q^i} \stackrel{\star}{=} (yq^n - t_1q^{n-1} - \dots - t_n)\right].$ 

L.H.S.:  $\left|q^n \sum_{n+1}^{\infty} \frac{t_i}{q^i}\right| \leq \sum_{1}^{\infty} \frac{\mathsf{const.}}{q^i} =: C_0.$ 

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$$\frac{1}{\left[\frac{t_1}{t_1} + \cdots + \frac{t_n}{t_n} + \sum_{n=1}^{\infty} \right]}$$

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, then  $\sigma_A(y)=\sum_{i=1}^\infty \frac{\sigma_A(t_i)}{A^i}$ .

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R.H.S.:

$$\left[\frac{t_1}{q} + \dots + \frac{t_n}{q^n} + \sum_{n+1}^{\infty} \right]$$

 $\left| \frac{t_1}{q} + \dots + \frac{t_n}{q^n} + \sum_{n+1}^{\infty} \frac{t_i}{q^i} = y \right| \Rightarrow \left[ q^n \sum_{n+1}^{\infty} \frac{t_i}{q^i} \stackrel{\star}{=} \left( yq^n - t_1q^{n-1} - \dots - t_n \right) =: \xi_n \right].$ 

L.H.S.:  $\left| q^n \sum_{n+1}^{\infty} \frac{t_i}{q^i} \right| \leq \sum_{1}^{\infty} \frac{\text{const.}}{q^i} =: C_0.$ 

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(b) If |A| > 1, then  $\sigma_A(y) = \sum_{i=1}^{\infty} \frac{\sigma_A(t_i)}{A^i}$ .

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R.H.S.:  $y \in Y = \bigcup_{\ell=0}^{\infty} \left[ q^{\ell}(\mathsf{D} - \mathsf{D}) + \dots + q(\mathsf{D} - \mathsf{D}) + (\mathsf{D} - \mathsf{D}) \right]$ 

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$$\left[\frac{\iota_1}{q} + \cdots + \frac{\iota_n}{q^n} + \sum_{n=0}^{\infty} \right]$$

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$$\left\lfloor \frac{\iota_1}{q} + \cdots + \frac{\iota_n}{q^n} + \sum_{r=1}^{n} \right\rfloor$$

$$\left[\frac{\iota_1}{q} + \cdots + \frac{\iota_n}{q^n} + \sum\right]$$

$$\begin{bmatrix} q & & & q^n & & \angle n \\ & & & & & \end{bmatrix}$$

L.H.S.: 
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 $\Rightarrow y \in \left[ q^{\ell}(\mathsf{D} - \mathsf{D}) + \dots + q(\mathsf{D} - \mathsf{D}) + (\mathsf{D} - \mathsf{D}) \right]$ 

$$(D) + \cdots + q(D-D) + (D-D)$$

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(b) If |A| > 1, then  $\sigma_A(y) = \sum_{i=1}^{\infty} \frac{\sigma_A(t_i)}{A^i}$ .

## **Proof of item(a)**

$$\left[\frac{t_1}{a} + \cdots + \frac{t_n}{a^n} + \sum\right]$$

 $\left| \frac{t_1}{q} + \dots + \frac{t_n}{q^n} + \sum_{n+1}^{\infty} \frac{t_i}{q^i} = y \right| \Rightarrow \left[ q^n \sum_{n+1}^{\infty} \frac{t_i}{q^i} \stackrel{\star}{=} \left( yq^n - t_1q^{n-1} - \dots - t_n \right) =: \xi_n \right].$ 

$$-\frac{t_n}{q^n} + \sum_{n+1}^{\infty}$$

$$+\frac{t_n}{a^n}+\sum_{i=1}^{\infty}$$

$$\frac{t_i}{q^i} = \sum_{n=1}^{\infty} \frac{t_i}{q^i} = \sum_{n=1}^{\infty} \frac{t_i}{q^i}$$

 $\Rightarrow y \in [q^{?}(D-D) + \cdots + q(D-D) + (D-D)]$ 

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R.H.S.:  $y \in Y = \bigcup_{\ell=0}^{\infty} \left[ q^{\ell}(\mathsf{D} - \mathsf{D}) + \dots + q(\mathsf{D} - \mathsf{D}) + (\mathsf{D} - \mathsf{D}) \right]$ 

$$\stackrel{\textstyle \smile}{=} n+1q$$

 $\Rightarrow \xi_n \in [q^{2+n}(D-D) + \cdots + q(D-D) + (D-D)]$  (because  $t_i \in (D-D)$ )

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H.S.:  $u \in Y = \prod_{n=1}^{\infty} \left[q^{\ell}(\mathsf{D}-\mathsf{D}) + \cdots + q(\mathsf{D}-\mathsf{D})\right]$ 

.H.S.: 
$$|q| \angle_{n+1} |\overline{q^i}| \le \angle_1 |\overline{q^i}| = 0.00$$
.

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R.H.S.: 
$$y \in Y = \bigcup_{\ell=0}^{\infty} \left[ q^{\ell}(D-D) + \dots + q(D-D) + (D-D) \right]$$

$$\left[q^{\ell}(\mathsf{D}-\mathsf{D})+\cdots+q(\mathsf{D}-\mathsf{D})+(\mathsf{D}-\mathsf{D})\right]$$

$$\underset{\ell=0}{\overset{\infty}{\triangleright}} \left[ q^{\ell}(\mathsf{D}-\mathsf{D}) + \dots + q(\mathsf{D}-\mathsf{D}) + (\mathsf{D}-\mathsf{D}) \right]$$

R.H.S.: 
$$y \in Y = \bigcup_{\ell=0} [q^{\ell}(D-D) + \dots + q(D-D) + (D-D)]$$
  
 $\Rightarrow y \in [q^{\ell}(D-D) + \dots + q(D-D) + (D-D)]$ 

$$\Rightarrow y \in [q^{?}(D-D) + \dots + q(D-D)]$$

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$$\Rightarrow y \in \left[q^{?}(\mathsf{D} - \mathsf{D}) + \dots + q(\mathsf{D} - \mathsf{D}) + (\mathsf{D} - \mathsf{D})\right]$$

$$\Rightarrow \xi \in \left[q^{?+n}(\mathsf{D} - \mathsf{D}) + \dots + q(\mathsf{D} - \mathsf{D}) + (\mathsf{D} - \mathsf{D})\right] \text{ (because } t \in (\mathsf{D} - \mathsf{D}))$$

$$\Rightarrow y \in \left[ q^{2}(\mathsf{D} - \mathsf{D}) + \dots + q(\mathsf{D} - \mathsf{D}) + (\mathsf{D} - \mathsf{D}) \right]$$

$$\Rightarrow y \in [q(D-D) + \dots + q(D-D) + (D-D)]$$
  
 
$$\Rightarrow \mathcal{E}_n \in [q^{2+n}(D-D) + \dots + q(D-D) + (D-D)] \text{ (because } t_i \in (D-D))$$

$$\Rightarrow \xi_n \in \left[q^{?+n}(\mathsf{D}-\mathsf{D}) + \dots + q(\mathsf{D}-\mathsf{D}) + (\mathsf{D}-\mathsf{D})\right] \text{ (because } t_i \in (\mathsf{D}-\mathsf{D})\text{)}$$

$$\Rightarrow \xi_n \in \left[q^{?+n}(\mathsf{D}-\mathsf{D}) + \dots + q(\mathsf{D}-\mathsf{D}) + (\mathsf{D}-\mathsf{D})\right] \text{ (because } t_i \in (\mathsf{D}-\mathsf{D})\text{)}$$

$$\Rightarrow \xi_n \in V$$
(D-D) + \cdots + q(D-D) + (D-D) \cdot (because  $t_i \in (D-D)$ )

$$\Rightarrow \xi_n \in [q \quad (B \quad B) + q(B \quad B) + (B \quad B)] \text{ (because } v_i \in (B \quad B))$$

$$\Rightarrow \xi_n \in Y.$$

FTC and  $y = \sum_{i=1}^{\infty} \frac{t_i}{a^i}$ , where  $t_i \in (D-D)$ . We have (a)  $\{yq^n - t_1q^{n-1} - \cdots - t_n : n \ge 0\}$  is a finite set. (b) If |A| > 1, then  $\sigma_A(y) = \sum_{i=1}^{\infty} \frac{\sigma_A(t_i)}{A^i}$ .

## **Proof of item(a)**

$$\left[\frac{t_1}{q} + \dots + \frac{t_n}{q^n} + \sum_{n+1}^{\infty} \frac{t_i}{q^i} = y\right] \Rightarrow \left[q^n \sum_{n+1}^{\infty} \frac{t_i}{q^i} \stackrel{\star}{=} (yq^n - t_1q^{n-1} - \dots - t_n) =: \xi_n\right].$$

L.H.S.:  $\left| q^n \sum_{n+1}^{\infty} \frac{t_i}{q^i} \right| \leq \sum_{1}^{\infty} \frac{\text{const.}}{q^i} =: C_0.$ 

H.S.: 
$$\left|q^{r^{\ell}}\sum_{n+1}\frac{q^{\ell}}{q^{\ell}}\right| \leq \sum_{1}\frac{\operatorname{const.}}{q^{\ell}} =: C_{0}.$$
H.S.:  $y \in Y = \bigcup_{\ell=0}^{\infty} \left[q^{\ell}(\mathsf{D}-\mathsf{D}) + \dots + q(\mathsf{D}-\mathsf{D})\right]$ 

R.H.S.:  $y \in Y = \bigcup_{\ell=0}^{\infty} \left[ q^{\ell}(\mathsf{D} - \mathsf{D}) + \dots + q(\mathsf{D} - \mathsf{D}) + (\mathsf{D} - \mathsf{D}) \right]$ 

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$$\Rightarrow y \in \left[ q^{2}(\mathsf{D} - \mathsf{D}) + \dots + q(\mathsf{D} - \mathsf{D}) + (\mathsf{D} - \mathsf{D}) \right]$$

$$\Rightarrow y \in \left[q^{?}(\mathsf{D}-\mathsf{D}) + \dots + q(\mathsf{D}-\mathsf{D}) + (\mathsf{D}-\mathsf{D})\right]$$
  

$$\Rightarrow \xi_{n} \in \left[q^{?+n}(\mathsf{D}-\mathsf{D}) + \dots + q(\mathsf{D}-\mathsf{D}) + (\mathsf{D}-\mathsf{D})\right] \text{ (because } t_{i} \in (\mathsf{D}-\mathsf{D})\text{)}$$

 $\Rightarrow \xi_n \in [q^{?+n}(D-D) + \cdots + q(D-D) + (D-D)]$  (because  $t_i \in (D-D)$ )

 $\Rightarrow \xi_n \in Y$ .

∴ L.H.S. & R.H.S.

FTC and  $y = \sum_{i=1}^{\infty} \frac{t_i}{a^i}$ , where  $t_i \in (D-D)$ . We have (a)  $\{yq^n - t_1q^{n-1} - \cdots - t_n : n \ge 0\}$  is a finite set.

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H.S.: 
$$y \in Y = \bigcup_{\ell=0}^{\infty} \left[ q^{\ell}(\mathsf{D} - \mathsf{D}) + \dots + q \right]$$
  
 $y \in \left[ q^{2}(\mathsf{D} - \mathsf{D}) + \dots + q(\mathsf{D} - \mathsf{D}) + (\mathsf{D} - \mathsf{D}) \right]$ 

 $\therefore$  L.H.S. & R.H.S.  $\Rightarrow \{\xi_n : n \geq 0\} \subseteq [-C_0, C_0] \cap Y$ .

R.H.S.: 
$$y \in Y = \bigcup_{\ell=0}^{\infty} \left[ q^{\ell}(\mathsf{D} - \mathsf{D}) + \dots + q(\mathsf{D} - \mathsf{D}) + (\mathsf{D} - \mathsf{D}) \right]$$
  

$$\Rightarrow y \in \left[ q^{2}(\mathsf{D} - \mathsf{D}) + \dots + q(\mathsf{D} - \mathsf{D}) + (\mathsf{D} - \mathsf{D}) \right]$$

$$\Rightarrow y \in \left[q^{?}(\mathsf{D}-\mathsf{D}) + \dots + q(\mathsf{D}-\mathsf{D}) + (\mathsf{D}-\mathsf{D})\right]$$

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$$\Rightarrow y \in [q (\mathsf{D} - \mathsf{D}) + \dots + q(\mathsf{D} - \mathsf{D}) + (\mathsf{D} - \mathsf{D})]$$

$$\Rightarrow \xi_n \in [q^{?+n}(\mathsf{D} - \mathsf{D}) + \dots + q(\mathsf{D} - \mathsf{D}) + (\mathsf{D} - \mathsf{D})] \text{ (because } t_i \in (\mathsf{D} - \mathsf{D}))$$

$$\Rightarrow y \in [q^{!}(\mathsf{D}-\mathsf{D}) + \dots + q(\mathsf{D}-\mathsf{D}) + (\mathsf{D}-\mathsf{D})]$$
  

$$\Rightarrow \xi_{n} \in [q^{?+n}(\mathsf{D}-\mathsf{D}) + \dots + q(\mathsf{D}-\mathsf{D}) + (\mathsf{D}-\mathsf{D})] \text{ (because } t_{i} \in (\mathsf{D}-\mathsf{D}))$$

$$\Rightarrow \xi_n \in \left[q^{t+n}(\mathsf{D}-\mathsf{D}) + \dots + q(\mathsf{D}-\mathsf{D}) + (\mathsf{D}-\mathsf{D})\right] \text{ (because } t_i \in (\mathsf{D}-\mathsf{D})\text{)}$$
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, then  $\sigma_A(y) = \sum_{i=1}^{\infty} \frac{\sigma_A(t_i)}{A^i}$ .

## **Proof of item(a)**

$$\begin{bmatrix} \frac{t_1}{q} + \dots + \frac{t_n}{q^n} + \sum_{n+1}^{\infty} \frac{t_i}{q^i} = y \end{bmatrix} \Rightarrow \begin{bmatrix} q^n \sum_{n+1}^{\infty} \frac{t_i}{q^i} \stackrel{\star}{=} (yq^n - t_1q^{n-1} - \dots - t_n) =: \xi_n \end{bmatrix}.$$

$$\mathsf{L.H.S.:} \quad \begin{vmatrix} q^n \sum_{n+1}^{\infty} \frac{t_i}{q^i} \end{vmatrix} \leq \sum_{1}^{\infty} \frac{\mathsf{const.}}{q^i} =: C_0.$$

R.H.S.:  $y \in Y = \bigcup_{\ell=0}^{\infty} \left[ q^{\ell}(\mathsf{D} - \mathsf{D}) + \dots + q(\mathsf{D} - \mathsf{D}) + (\mathsf{D} - \mathsf{D}) \right]$ 

R.H.S.: 
$$y \in Y = \bigcup_{\ell=0} [q^{\ell}(D-D) + \dots + q(D-D) + (D-D)]$$
  
 $\Rightarrow y \in [q^{\ell}(D-D) + \dots + q(D-D) + (D-D)]$   
 $\Rightarrow \xi_n \in [q^{\ell+n}(D-D) + \dots + q(D-D) + (D-D)]$  (because  $t_i \in (D-D)$ )

 $\Rightarrow \xi_n \in Y$ .

$$\therefore$$
 L.H.S. & R.H.S.  $\Rightarrow \{\xi_n : n \geq 0\} \subseteq [-C_0, C_0] \cap Y$ .

 $:: \mathsf{FTC} \Rightarrow Y \mathsf{has} \mathsf{no} \mathsf{accumulation} \mathsf{point}$ 

FTC and  $y = \sum_{i=1}^{\infty} \frac{t_i}{a^i}$ , where  $t_i \in (D-D)$ . We have (a)  $\{yq^n - t_1q^{n-1} - \cdots - t_n : n \ge 0\}$  is a finite set.

(b) If |A| > 1, then  $\sigma_A(y) = \sum_{i=1}^{\infty} \frac{\sigma_A(t_i)}{A_i}$ .

#### Proof of item(a)

 $\left| \frac{t_1}{q} + \dots + \frac{t_n}{q^n} + \sum_{n+1}^{\infty} \frac{t_i}{q^i} = y \right| \Rightarrow \left[ q^n \sum_{n+1}^{\infty} \frac{t_i}{q^i} \stackrel{\star}{=} (yq^n - t_1q^{n-1} - \dots - t_n) =: \xi_n \right].$ 

L.H.S.: 
$$\left| q^n \sum_{n+1}^{\infty} \frac{t_i}{q^i} \right| \leq \sum_{1}^{\infty} \frac{\text{const.}}{q^i} =: C_0.$$

R.H.S.:  $y \in Y = \bigcup_{\ell=0}^{\infty} \left[ q^{\ell}(\mathsf{D} - \mathsf{D}) + \dots + q(\mathsf{D} - \mathsf{D}) + (\mathsf{D} - \mathsf{D}) \right]$ 

R.H.S.: 
$$y \in Y = \bigcup_{\ell=0}^{\infty} \left[ q^{\ell}(\mathsf{D} - \mathsf{D}) + \dots + q(\mathsf{I}) \right]$$
  
 $\Rightarrow y \in \left[ q^{2}(\mathsf{D} - \mathsf{D}) + \dots + q(\mathsf{D} - \mathsf{D}) + (\mathsf{D} - \mathsf{D}) \right]$ 

 $\Rightarrow \xi_n \in Y$ .

 $\Rightarrow \xi_n \in [q^{?+n}(D-D) + \cdots + q(D-D) + (D-D)]$  (because  $t_i \in (D-D)$ )

 $\therefore$  L.H.S. & R.H.S.  $\Rightarrow \{\xi_n : n \geq 0\} \subseteq [-C_0, C_0] \cap Y$ .  $:: \mathsf{FTC} \Rightarrow Y \text{ has no accumulation point } \Rightarrow [-C_0, C_0] \cap Y \text{ is a finite set}$ 

FTC and  $y = \sum_{i=1}^{\infty} \frac{t_i}{a^i}$ , where  $t_i \in (D-D)$ . We have (a)  $\{yq^n - t_1q^{n-1} - \cdots - t_n : n \ge 0\}$  is a finite set.

(b) If 
$$|A| > 1$$
, then  $\sigma_A(y) = \sum_{i=1}^{\infty} \frac{\sigma_A(t_i)}{A^i}$ .

#### Proof of item(a)

$$\begin{bmatrix} \frac{t_1}{q} + \dots + \frac{t_n}{q^n} + \sum_{n+1}^{\infty} \frac{t_i}{q^i} = y \end{bmatrix} \Rightarrow \begin{bmatrix} q^n \sum_{n+1}^{\infty} \frac{t_i}{q^i} \stackrel{\star}{=} (yq^n - t_1q^{n-1} - \dots - t_n) =: \xi_n \end{bmatrix}.$$

$$\mathsf{L.H.S.:} \quad \begin{vmatrix} q^n \sum_{n+1}^{\infty} \frac{t_i}{q^i} \end{vmatrix} \leq \sum_{1}^{\infty} \frac{\mathsf{const.}}{q^i} =: C_0.$$

R.H.S.:  $y \in Y = \bigcup_{\ell=0}^{\infty} \left[ q^{\ell}(\mathsf{D} - \mathsf{D}) + \dots + q(\mathsf{D} - \mathsf{D}) + (\mathsf{D} - \mathsf{D}) \right]$ 

$$\Rightarrow y \in \left[q^{?}(\mathsf{D}-\mathsf{D}) + \dots + q(\mathsf{D}-\mathsf{D}) + (\mathsf{D}-\mathsf{D})\right]$$

 $\Rightarrow \xi_n \in [q^{?+n}(D-D) + \cdots + q(D-D) + (D-D)]$  (because  $t_i \in (D-D)$ )  $\Rightarrow \xi_n \in Y$ .

 $\therefore$  L.H.S. & R.H.S.  $\Rightarrow \{\xi_n : n \geq 0\} \subseteq [-C_0, C_0] \cap Y$ .

 $:: \mathsf{FTC} \Rightarrow Y \text{ has no accumulation point } \Rightarrow [-C_0, C_0] \cap Y \text{ is a finite set}$  $\therefore \{\xi_n : n \geq 0\}$  is also a finite set.

FTC and  $y = \sum_{i=1}^{\infty} \frac{t_i}{a^i}$ , where  $t_i \in (D-D)$ . We have (a)  $\{yq^n - t_1q^{n-1} - \cdots - t_n : n \ge 0\}$  is a finite set.

(b) If |A| > 1, then  $\sigma_A(y) = \sum_{i=1}^{\infty} \frac{\sigma_A(t_i)}{A^i}$ .

## Proof of item(a)

 $\left| \frac{t_1}{q} + \dots + \frac{t_n}{q^n} + \sum_{n+1}^{\infty} \frac{t_i}{q^i} = y \right| \Rightarrow \left[ q^n \sum_{n+1}^{\infty} \frac{t_i}{q^i} \stackrel{\star}{=} \left( yq^n - t_1q^{n-1} - \dots - t_n \right) =: \xi_n \right].$ 

$$\begin{bmatrix} q & & & q^n & & \\ & & & & \\ & & & \end{bmatrix} q^n \sum_{n=1}^{\infty}$$

L.H.S.:  $\left| q^n \sum_{n+1}^{\infty} \frac{t_i}{q^i} \right| \leq \sum_{1}^{\infty} \frac{\text{const.}}{q^i} =: C_0.$ 

 $\Rightarrow y \in [q^{?}(D-D) + \cdots + q(D-D) + (D-D)]$ 

R.H.S.: 
$$y \in Y = \bigcup_{\ell=0}^{\infty} \left[ q^{\ell}(D-D) + \dots + q(D-D) + (D-D) \right]$$

 $\Rightarrow \xi_n \in [q^{2+n}(D-D) + \cdots + q(D-D) + (D-D)]$  (because  $t_i \in (D-D)$ )

 $\Rightarrow \xi_n \in Y$ .

This completes the proof of item(a).

 $\therefore$  L.H.S. & R.H.S.  $\Rightarrow \{\xi_n : n \geq 0\} \subseteq [-C_0, C_0] \cap Y$ . :: FTC  $\Rightarrow Y$  has no accumulation point  $\Rightarrow [-C_0, C_0] \cap Y$  is a finite set

 $\therefore \{\xi_n : n \ge 0\}$  is also a finite set.

(a)  $\{yq^n - t_1q^{n-1} - \cdots - t_n : n \ge 0\}$  is a finite set. (b) If |A| > 1, then  $\sigma_A(y) = \sum_{i=1}^{\infty} \frac{\sigma_A(t_i)}{A_i}$ .

FTC and  $y = \sum_{i=1}^{\infty} \frac{t_i}{a^i}$ , where  $t_i \in (D-D)$ . We have

## Proof of item(a)

 $\left| \frac{t_1}{q} + \dots + \frac{t_n}{q^n} + \sum_{n+1}^{\infty} \frac{t_i}{q^i} = y \right| \Rightarrow \left| q^n \sum_{n+1}^{\infty} \frac{t_i}{q^i} \stackrel{\star}{=} (yq^n - t_1q^{n-1} - \dots - t_n) =: \xi_n \right|.$ 

L.H.S.:  $\left| q^n \sum_{n+1}^{\infty} \frac{t_i}{q^i} \right| \leq \sum_{1}^{\infty} \frac{\text{const.}}{q^i} =: C_0.$ 

R.H.S.:  $y \in Y = \bigcup_{\ell=0}^{\infty} \left[ q^{\ell}(\mathsf{D} - \mathsf{D}) + \dots + q(\mathsf{D} - \mathsf{D}) + (\mathsf{D} - \mathsf{D}) \right]$ 

 $\Rightarrow y \in [q^{?}(D-D) + \cdots + q(D-D) + (D-D)]$ 

 $\Rightarrow \xi_n \in [q^{2+n}(D-D) + \cdots + q(D-D) + (D-D)]$  (because  $t_i \in (D-D)$ )  $\Rightarrow \xi_n \in Y$ .

 $\therefore$  L.H.S. & R.H.S.  $\Rightarrow \{\xi_n : n \geq 0\} \subseteq [-C_0, C_0] \cap Y$ .

:: FTC  $\Rightarrow Y$  has no accumulation point  $\Rightarrow [-C_0, C_0] \cap Y$  is a finite set

 $\therefore \{\xi_n : n \ge 0\}$  is also a finite set.

This completes the proof of item(a).

FTC and  $y = \sum_{i=1}^{\infty} \frac{t_i}{a^i}$ , where  $t_i \in (D-D)$ . We have

(a) 
$$\{yq^n - t_1q^{n-1} - \cdots - t_n : n \ge 0\}$$
 is a finite set.  
(b) If  $|A| > 1$ , then  $\sigma_A(y) = \sum_{i=1}^{\infty} \frac{\sigma_A(t_i)}{A^i}$ .

## Proof of [item(a) $\Rightarrow$ item(b)]

FTC and  $y = \sum_{i=1}^{\infty} \frac{t_i}{q^i}$ , where  $t_i \in (D-D)$ . We have (a)  $\{yq^n - t_1q^{n-1} - \cdots - t_n : n \geq 0\}$  is a finite set.

(b) If 
$$|A|>1$$
, then  $\sigma_A(y)=\sum_{i=1}^\infty \frac{\sigma_A(t_i)}{A^i}$ .

#### Proof of [item(a) $\Rightarrow$ item(b)]

$$\frac{Yq^n - t_1q^{n-1} - \dots - t_n : n \ge 0}{\{yq^n - t_1q^{n-1} - \dots - t_n : n \ge 0\}}$$
 is a finite set

FTC and  $y = \sum_{i=1}^{\infty} \frac{t_i}{a^i}$ , where  $t_i \in (D-D)$ . We have (a)  $\{yq^n - t_1q^{n-1} - \cdots - t_n : n \ge 0\}$  is a finite set.

(b) If 
$$|A|>1$$
, then  $\sigma_A(y)=\sum_{i=1}^\infty \frac{\sigma_A(t_i)}{A^i}$ .

 $\Rightarrow \{\sigma_A(yq^n - t_1q^{n-1} - \cdots - t_n) : n \ge 0\}$  is a finite set

## Proof of [item(a) $\Rightarrow$ item(b)]

$$\{yq^n-t_1q^{n-1}-\cdots-t_n:\ n\geq 0\}$$
 is a finite set

FTC and  $y = \sum_{i=1}^{\infty} \frac{t_i}{a^i}$ , where  $t_i \in (D-D)$ . We have (a)  $\{yq^n - t_1q^{n-1} - \cdots - t_n : n \ge 0\}$  is a finite set.

(b) If 
$$|A| > 1$$
, then  $\sigma_A(y) = \sum_{i=1}^{\infty} \frac{\sigma_A(t_i)}{A^i}$ .

 $\Rightarrow \{\sigma_A(yq^n-t_1q^{n-1}-\cdots-t_n): n\geq 0\}$  is a finite set

## **Proof of [item(a)** $\Rightarrow$ item(b)]

$$\{yq^n-t_1q^{n-1}-\cdots-t_n:\ n\geq 0\}$$
 is a finite set

 $\Rightarrow |\sigma_A(yq^n - t_1q^{n-1} - \cdots - t_n)| < \text{const.}$ 

FTC and  $y = \sum_{i=1}^{\infty} \frac{t_i}{a^i}$ , where  $t_i \in (D-D)$ . We have (a)  $\{yq^n - t_1q^{n-1} - \dots - t_n : n \ge 0\}$  is a finite set.

(b) If 
$$|A| > 1$$
, then  $\sigma_A(y) = \sum_{i=1}^{\infty} \frac{\sigma_A(t_i)}{A^i}$ .

#### **Proof of [item(a)** $\Rightarrow$ item(b)]

$$\int ya^n - t \cdot a^{n-1} - \cdots - t \cdot n > 0$$

$$\{yq^n-t_1q^{n-1}-\cdots-t_n: n\geq 0\}$$
 is a finite set

$$\{yq^n - t_1q^{n-1} - \dots - t_n : n \ge 1\}$$

$$yq^n - t_1q^{n-1} - \cdots - t_n: n \ge$$

$$\{yq^n - t_1q^{n-1} - \dots - t_n : n \ge 0\}$$
 is a finite set  $\Rightarrow \{\sigma_A(yq^n - t_1q^{n-1} - \dots - t_n) : n \ge 0\}$  is a finite set

 $\Rightarrow |\sigma_A(yq^n - t_1q^{n-1} - \cdots - t_n)| \leq \text{const.}$ 

Therefore,  $\left|\sigma_A(y) - \frac{\sigma_A(t_1)}{A} - \cdots - \frac{\sigma_A(t_n)}{A^n}\right|$ 

FTC and  $y = \sum_{i=1}^{\infty} \frac{t_i}{a^i}$ , where  $t_i \in (D-D)$ . We have (a)  $\{yq^n - t_1q^{n-1} - \cdots - t_n : n \ge 0\}$  is a finite set.

(b) If 
$$|A|>1$$
, then  $\sigma_A(y)=\sum_{i=1}^\infty \frac{\sigma_A(t_i)}{A^i}$ .

#### Proof of [item(a) $\Rightarrow$ item(b)]

$$\{yq^n-t_1q^{n-1}-\cdots-t_n:\ n\geq 0\}$$
 is a finite set

$$yq^n - t_1q^{n-1} - \dots - t_n: n \ge 0$$

$$yq^n - t_1q^{n-1} - \dots - t_n: n \ge 0$$

$$yq^n - t_1q^{n-1} - \dots - t_n: n \ge 1$$

$$\begin{cases} yq - t_1q - \cdots - t_n : n \ge 0 \end{cases} \text{ is a finite set}$$

$$\Rightarrow \left\{ \sigma_A(yq^n - t_1q^{n-1} - \cdots - t_n) : n \ge 0 \right\} \text{ is a finite set}$$

$$\Rightarrow \left| \sigma_A(yq^n - t_1q^{n-1} - \cdots - t_n) \right| \le \text{const.}$$

Therefore,  $\left| \sigma_A(y) - \frac{\sigma_A(t_1)}{A} - \dots - \frac{\sigma_A(t_n)}{A^n} \right|$   $= \left| \frac{\sigma_A(yq^n - t_1q^{n-1} - \dots - t_n)}{A^n} \right|$ 

FTC and  $y = \sum_{i=1}^{\infty} \frac{t_i}{a^i}$ , where  $t_i \in (D-D)$ . We have (a)  $\{yq^n - t_1q^{n-1} - \cdots - t_n : n \ge 0\}$  is a finite set.

(b) If 
$$|A| > 1$$
, then  $\sigma_A(y) = \sum_{i=1}^{\infty} \frac{\sigma_A(t_i)}{A^i}$ .

### Proof of [item(a) $\Rightarrow$ item(b)]

$$\{yq^n-t_1q^{n-1}-\cdots-t_n:\ n\geq 0\}$$
 is a finite set

$$yq^n - t_1q^{n-1} - \dots - t_n: n \ge 0$$

$$yq^n - t_1q^{n-1} - \dots - t_n: n \ge 0$$

$$\cdots - t_n: n \geq 0$$
 $n-1 - \cdots - t_m$ 

$$\cdots - t_n: n \ge 0$$

 $\Rightarrow \{\sigma_A(yq^n-t_1q^{n-1}-\cdots-t_n): n\geq 0\}$  is a finite set

$$n \geq 0$$
  $\cdots - t_n$ :

 $\Rightarrow \left| \frac{\sigma_A(yq^n - t_1q^{n-1} - \dots - t_n)}{\sigma_A(yq^n - t_1q^{n-1} - \dots - t_n)} \right| \le \text{const.}$ Therefore,  $\left| \frac{\sigma_A(yq^n - t_1q^{n-1} - \dots - t_n)}{A} \right| = \left| \frac{\sigma_A(yq^n - t_1q^{n-1} - \dots - t_n)}{A^n} \right| \le \frac{\text{const.}}{|A|^n}$ 

$$s = finite$$
  
 $s > 0$  is

FTC and  $y = \sum_{i=1}^{\infty} \frac{t_i}{a^i}$ , where  $t_i \in (D-D)$ . We have (a)  $\{yq^n - t_1q^{n-1} - \cdots - t_n : n \ge 0\}$  is a finite set.

 $= \left| \frac{\sigma_A(yq^n - t_1q^{n-1} - \dots - t_n)}{A^n} \right| \leq \frac{\mathsf{const.}}{\left|A\right|^n} \to 0 \text{ as } n \to \infty \text{ (by } |A| > 1).$ 

(b) If |A| > 1, then  $\sigma_A(y) = \sum_{i=1}^{\infty} \frac{\sigma_A(t_i)}{A_i}$ .

#### Proof of [item(a) $\Rightarrow$ item(b)]

$$yq^n - t_1q^{n-1} - \dots - t_n: n \ge 1$$

$$\left\{yq^n-t_1q^{n-1}-\cdots-t_n:\ n\geq 0\right\}$$
 is a finite set

 $\Rightarrow \{\sigma_A(yq^n-t_1q^{n-1}-\cdots-t_n): n\geq 0\}$  is a finite set

 $\Rightarrow \left|\sigma_A(yq^n - t_1q^{n-1} - \dots - t_n)\right| \leq \text{const.}$ 

Therefore,  $\left| \sigma_A(y) - \frac{\sigma_A(t_1)}{A} - \dots - \frac{\sigma_A(t_n)}{A^n} \right|$ 

FTC and  $y = \sum_{i=1}^{\infty} \frac{t_i}{a^i}$ , where  $t_i \in (D-D)$ . We have (a)  $\{yq^n - t_1q^{n-1} - \cdots - t_n : n \ge 0\}$  is a finite set.

(b) If 
$$|A| > 1$$
, then  $\sigma_A(y) = \sum_{i=1}^{\infty} \frac{\sigma_A(t_i)}{A^i}$ .

# **Proof of [item(a)** $\Rightarrow$ item(b)]

$$\{yq^n - t_1q^{n-1} - \dots - t_n : n \ge 0\} \text{ is a finite set}$$

 $\Rightarrow \{\sigma_A(yq^n-t_1q^{n-1}-\cdots-t_n): n\geq 0\}$  is a finite set

$$\{\sigma_A(yq^n-t_1q^{n-1}-\cdots-t_n):\ n\geq 0\}$$
 is a form  $\{\sigma_A(yq^n-t_1q^{n-1}-\cdots-t_n)\}$ 

$$\{\sigma_A(yq^n-t_1q^{n-1}-\cdots-t_n):\ n\geq 0\}$$
 is a f $|\sigma_A(yq^n-t_1q^{n-1}-\cdots-t_n)|<\mathsf{const.}$ 

$$\Rightarrow |\sigma_A(yq^n - t_1q^{n-1} - \dots - t_n)| \le \text{const.}$$

$$\Rightarrow |\sigma_A(yq^n - t_1q^{n-1} - \dots - t_n)| \le \sigma_A(t_n)$$

$$\left|\sigma_A(yq^n-t_1q^{n-1}-\cdots-t_n)
ight| \leq ext{const.}$$
 erefore,  $\left|\sigma_A(y)-rac{\sigma_A(t_1)}{t}-\cdots-rac{\sigma_A(t_n)}{t}
ight|$ 

Therefore, 
$$\left| \sigma_A(y) - \frac{\sigma_A(t_1)}{A} - \dots - \frac{\sigma_A(t_n)}{A^n} \right|$$

Therefore, 
$$\left| \sigma_A(y) - \frac{\sigma_A(t_1)}{A} - \dots - \frac{\sigma_A(t_n)}{A^n} \right|$$

Therefore, 
$$\left| \sigma_A(y) - \frac{\sigma_A(v_1)}{A} - \dots - \frac{\sigma_A(v_n)}{A^n} \right|$$
 $\left| \sigma_A(yq^n - t_1q^{n-1} - \dots - t_n) \right| \leq \text{const.}$ 

$$= \left|\frac{\sigma_A(yq^n - t_1q^{n-1} - \dots - t_n)}{A^n}\right| \leq \frac{\mathsf{const.}}{|A|^n} \to 0 \text{ as } n \to \infty \text{ (by } |A| > 1).$$

$$= \left| \frac{\sigma_A(yq^n - t_1q^{n-1} - \dots - t_n)}{A^n} \right| \le \frac{\mathsf{const.}}{\left|A\right|^n} \to 0 \text{ as } n \to \infty \text{ (by } |A| > 1).$$

$$= \left| \frac{\sigma_A(yq^n - t_1q^{n-1} - \dots - t_n)}{A^n} \right| \le \frac{\text{const.}}{|A|^n} \to 0 \text{ as } n \to \infty \text{ (by } |A| > 1).$$
Hence  $\sigma_A(y) = \frac{\sigma_A(t_1)}{A} + \frac{\sigma_A(t_2)}{A^2} + \dots$ 

FTC and  $y = \sum_{i=1}^{\infty} \frac{t_i}{a^i}$ , where  $t_i \in (D-D)$ . We have (a)  $\{yq^n - t_1q^{n-1} - \cdots - t_n : n \ge 0\}$  is a finite set.

(b) If 
$$|A|>1$$
, then  $\sigma_A(y)=\sum_{i=1}^{\infty}\frac{\sigma_A(t_i)}{A^i}$ .

### **Proof of [item(a)** $\Rightarrow$ item(b)]

$$\{yq^n-t_1q^{n-1}-\cdots-t_n:\ n\geq 0\}$$
 is a finite set

$$\{yq^n-t_1q^{n-1}-\cdots-t_n:\ n\geq 0\}$$
 is a finite se

$$\forall q \quad t_1 q \quad t_n \cdot h \ge 0$$
 is a finite set  $\forall q \quad t_n \cdot h \ge 0$  is a finite set  $\forall q \quad t_1 q^n = t_1 q^{n-1} = \dots = t$ .

$$\Rightarrow \left\{ \sigma_A(yq^n - t_1q^{n-1} - \dots - t_n) : n \ge 0 \right\} \text{ is a finite set}$$

$$\left\{ \sigma_A(yq^n - t_1q^{n-1} - \dots - t_n) : n \ge 0 \right\} \text{ is a}$$
$$\left| \sigma_A(yq^n - t_1q^{n-1} - \dots - t_n) \right| < \text{const.}$$

$$\Rightarrow \left| \sigma_A(yq^n - t_1q^{n-1} - \dots - t_n) \right| \cdot n \ge 0$$
 is a 
$$\Rightarrow \left| \sigma_A(yq^n - t_1q^{n-1} - \dots - t_n) \right| \le \text{const.}$$

herefore, 
$$\left| \sigma_A(y) - \frac{\sigma_A(t_1)}{A} - \dots - \frac{\sigma_A(t_n)}{A^n} \right|$$
  
 $\left| \sigma_A(yq^n - t_1q^{n-1} - \dots - t_n) \right|$  const.

Therefore, 
$$\left|\sigma_A(y) - \frac{\sigma_A(t_1)}{A} - \dots - \frac{\sigma_A(t_n)}{A^n}\right|$$

Hence 
$$\sigma_A(y) = \frac{\sigma_A(t_1)}{A} + \frac{\sigma_A(t_2)}{A^2} + \cdots$$

Therefore, 
$$\left| \sigma_A(y) - \frac{\sigma_A(t_1)}{A} - \dots - \frac{\sigma_A(t_n)}{A^n} \right|$$
  
=  $\left| \frac{\sigma_A(yq^n - t_1q^{n-1} - \dots - t_n)}{A^n} \right| \le \frac{\text{const.}}{\left| A \right|^n} \to 0 \text{ as } n \to \infty \text{ (by } |A| > 1).$ 

Q.E.D.

FTC and  $y = \sum_{i=1}^{\infty} \frac{t_i}{a^i}$ , where  $t_i \in (D-D)$ . We have (a)  $\{yq^n - t_1q^{n-1} - \cdots - t_n : n \ge 0\}$  is a finite set.

(b) If 
$$|A| > 1$$
, then  $\sigma_A(y) = \sum_{i=1}^{\infty} \frac{\sigma_A(t_i)}{A^i}$ .

### **Proof of [item(a)** $\Rightarrow$ item(b)]

$$\int u e^{n} + e^{n-1} + \dots$$

 $\{yq^n-t_1q^{n-1}-\cdots-t_n: n>0\}$  is a finite set  $\Rightarrow \{\sigma_A(yq^n-t_1q^{n-1}-\cdots-t_n): n\geq 0\}$  is a finite set

$$\left\{\sigma_A(yq^n-t_1q^{n-1}-\cdots-t_n):\ n\geq 0\right\}$$
 is a

$$\{\sigma_A(yq^n - t_1q^{n-1} - \dots - t_n) : n \ge 0\}$$
 is a  $|\sigma_A(yq^n - t_1q^{n-1} - \dots - t_n)| < \text{const.}$ 

$$\Rightarrow \left| \sigma_A(yq^n - t_1q^{n-1} - \dots - t_n) \right| \leq \text{const.}$$

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Therefore, 
$$\left|\sigma_A(y) - \frac{\sigma_A(t_1)}{A} - \dots - \frac{\sigma_A(t_n)}{A^n}\right|$$

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 $\left|\sigma_A(yq^n - t_1q^{n-1} - \dots - t_n)\right|$  const.

Therefore, 
$$\left|\sigma_A(y) - \frac{1}{A} - \dots - \frac{1}{A^n}\right|$$

$$= \left|\frac{\sigma_A(yq^n - t_1q^{n-1} - \dots - t_n)}{A}\right| < \frac{\mathsf{const.}}{2} \to 0 \text{ as } n \to \infty \text{ (by } |A| > 1)$$

Therefore, 
$$\left| \frac{\sigma_A(y) - \overline{A} - \cdots - \overline{A^n}}{A} \right|$$

$$= \left| \frac{\sigma_A(yq^n - t_1q^{n-1} - \cdots - t_n)}{A^n} \right| \leq \frac{\mathsf{const.}}{|A|^n} \to 0 \text{ as } n \to \infty \text{ (by } |A| > 1).$$

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Hence  $\sigma_A(y) = \frac{\sigma_A(t_1)}{4} + \frac{\sigma_A(t_2)}{42} + \cdots$ .

Q.E.D.

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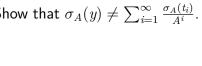
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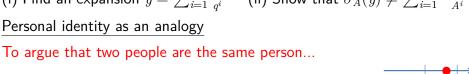
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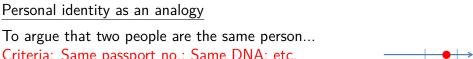




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Criteria: Same passport no.; Same DNA; etc.

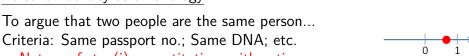


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Personal identity as an analogy

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Personal identity as an analogy

To argue that two people are the same person...

Criteria: Same passport no.; Same DNA; etc.

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To argue that two people are NOT the same person...









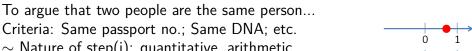
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# Proposition 1 (Consequence of FTC)

Let A be an algebraic conjugate of q, and  $y \in Y$ . Suppose  $\Phi$  satisfies the FTC and  $y = \sum_{i=1}^{\infty} \frac{t_i}{q^i}$ , where  $t_i \in (\mathsf{D} - \mathsf{D})$ . If |A| > 1, then  $\sigma_A(y) = \sum_{i=1}^{\infty} \frac{\sigma_A(t_i)}{A^i}$ .

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... The geometry of  $\{\sigma_A(\partial_i)\}_{i=0}^m$  plays a role in this problem.

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Method of step(i): a lazy algorithm

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#### Electric switch box as an analogy...



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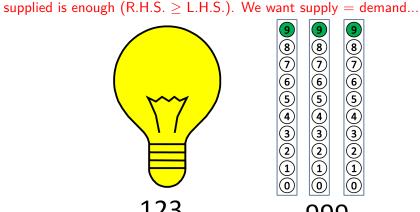
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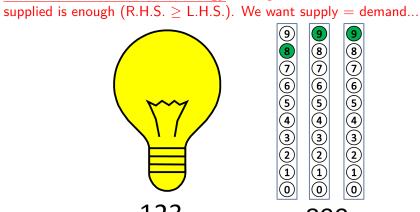
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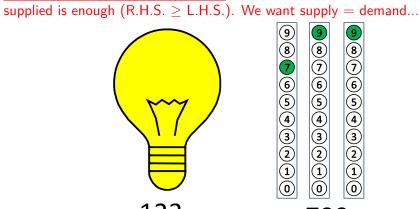
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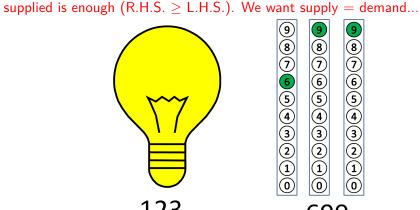
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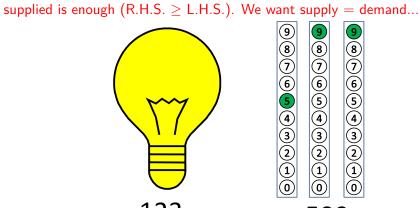
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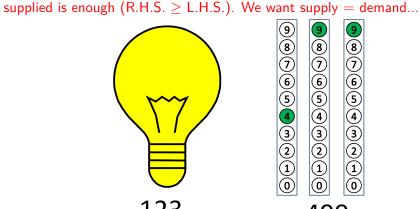
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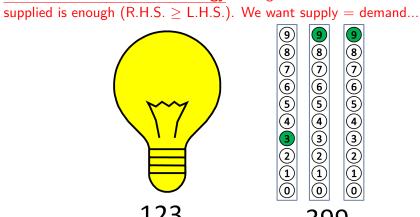
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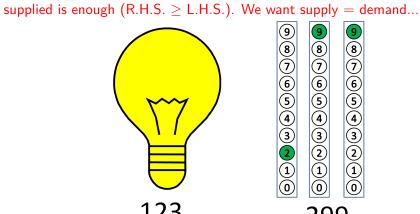
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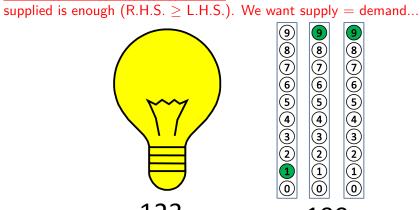
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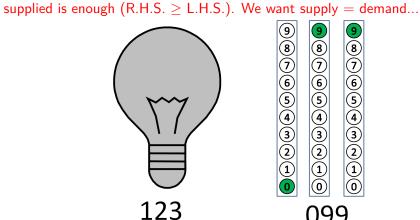
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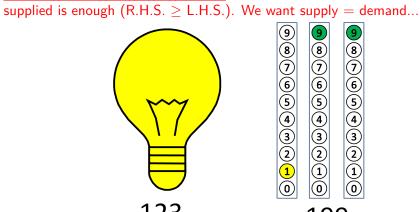
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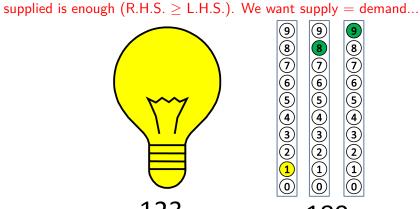
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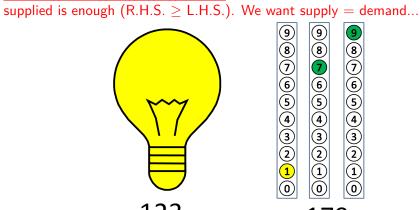
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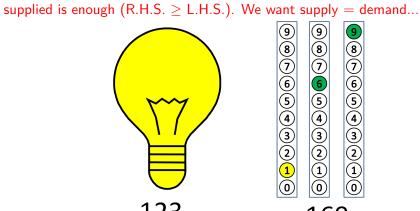
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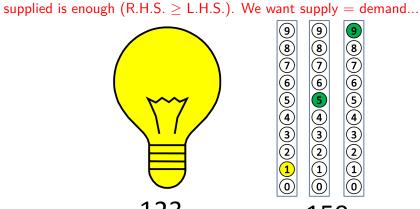
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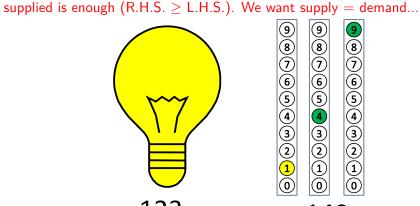


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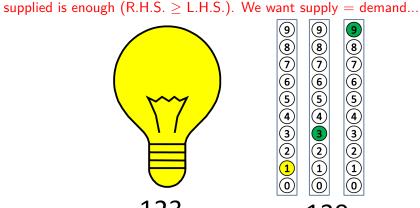
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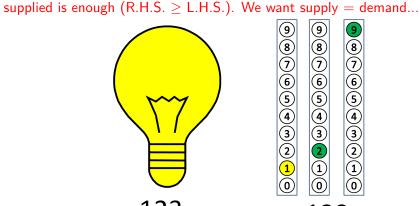
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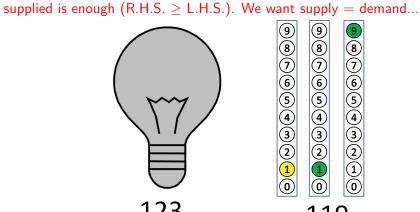
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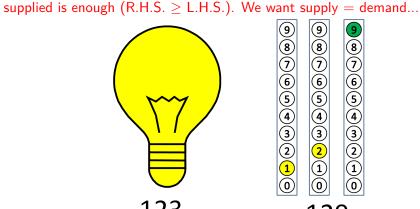
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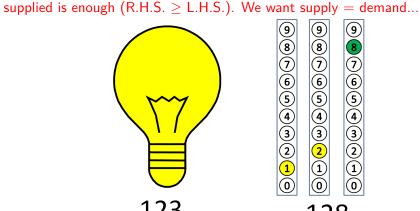
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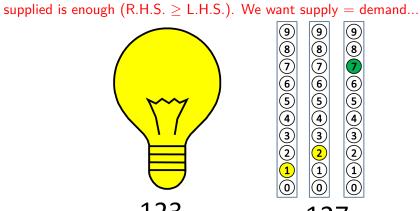
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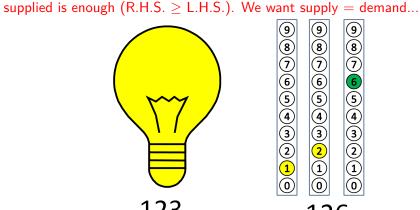
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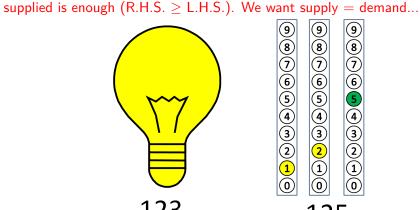
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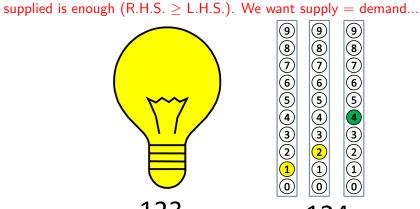
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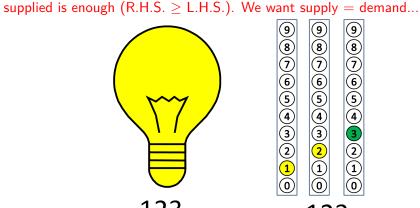
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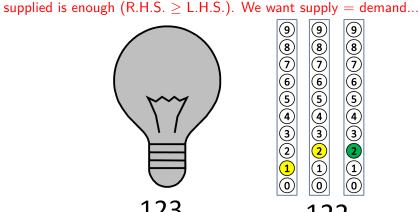
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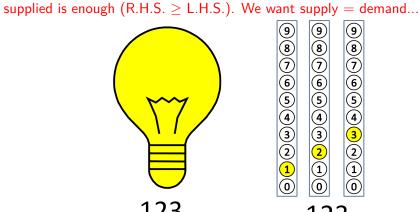
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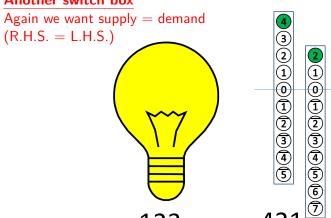
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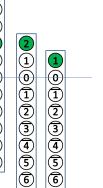
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Another switch box

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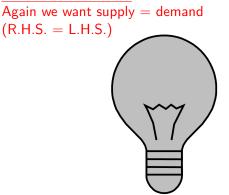


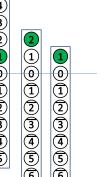


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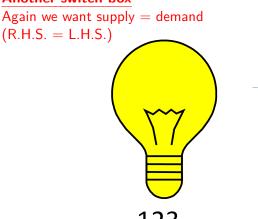




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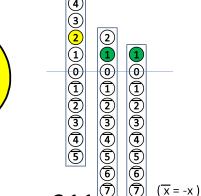
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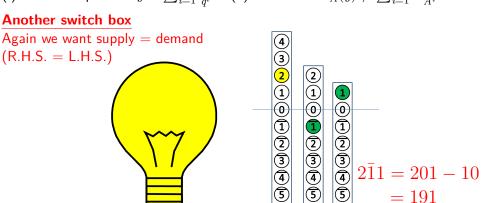
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### Another switch b

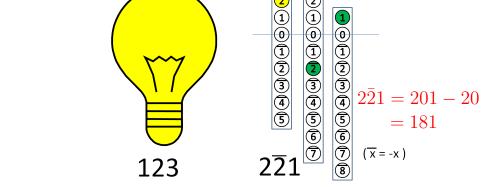




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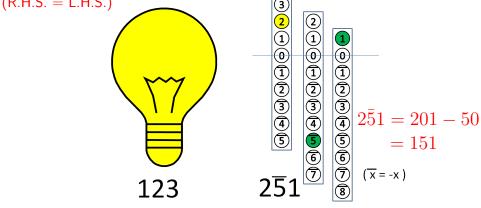
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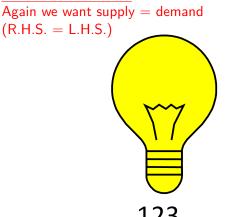
 $2\overline{7}1 = 201 - 70$ 

= 131

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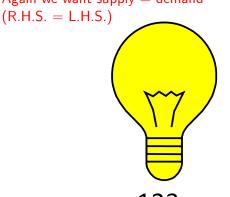
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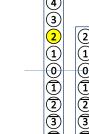
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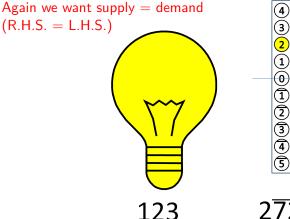
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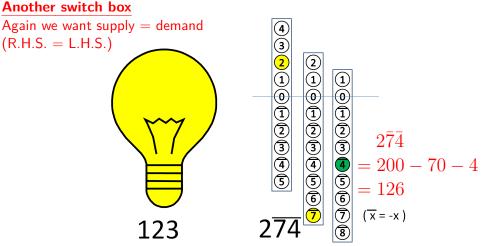


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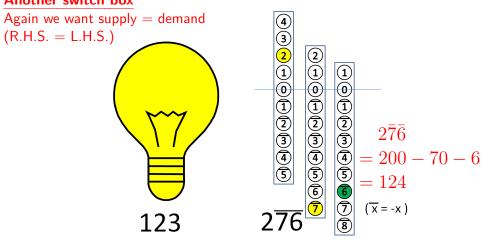
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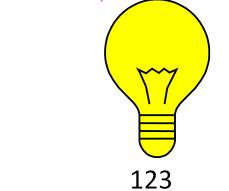
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$$|A|$$

$$\begin{array}{c|c} \hline 1 \\ \hline 0 \\ \hline \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline = 200 - 70 - 5 \\ \hline 5 \\ \hline 6 \\ \hline 7 \\ \hline \hline 8 \\ \hline \end{array}$$

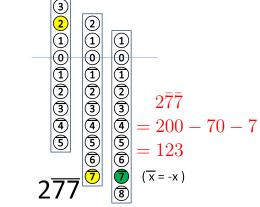
## **Proposition 1 (Consequence of FTC)** Let A be an algebraic conjugate of q, and $y \in Y$ . Suppose $\Phi$ satisfies the FTC and $y = \sum_{i=1}^{\infty} \frac{t_i}{\sigma^i}$ , where $t_i \in (D-D)$ . If |A| > 1, then $\sigma_A(y) = \sum_{i=1}^{\infty} \frac{\sigma_A(t_i)}{A^i}$ . It gives a two-step strategy of showing $|A| \leq 1$ : (i) Find an expansion $y = \sum_{i=1}^{\infty} \frac{t_i}{\sigma^i}$ (ii) Show that $\sigma_A(y) \neq \sum_{i=1}^{\infty} \frac{\sigma_A(t_i)}{\Lambda^i}$ .





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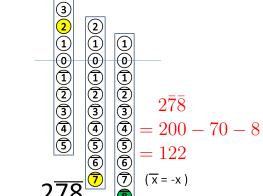
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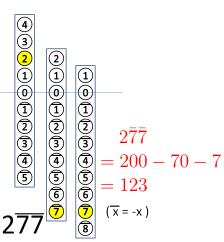
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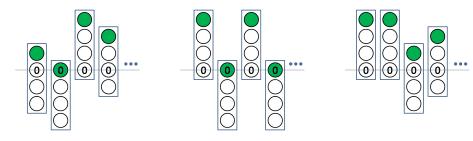
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### Lazy algorithm for step(i)

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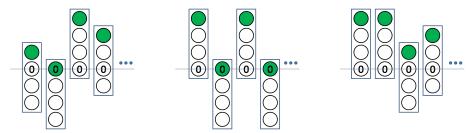
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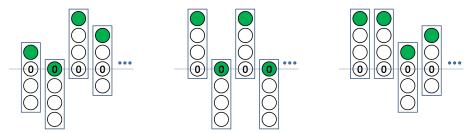
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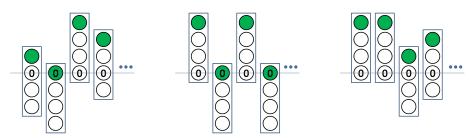
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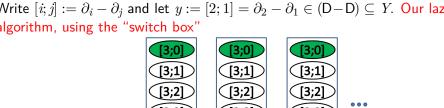
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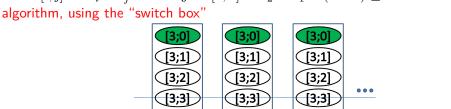
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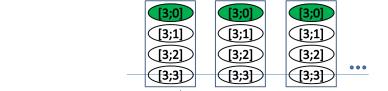
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Write  $[i,j] := \partial_i - \partial_j$  and let  $y := [2,1] = \partial_2 - \partial_1 \in (D-D) \subseteq Y$ . Our lazy algorithm, using the "switch box"

[3:1] [[3;1] [3;1] [3:2] [3;2] [3;2] [3:3] [3:3]

$$\begin{array}{c|cccc}
\hline
[3;1] & \hline
[3;1] & \hline
[3;1] & \hline
[3;2] & \hline
[$$

gives an expansion  $y = \sum_{i=1}^{\infty} \frac{t_i}{a^i}$ , where  $t_i \in \{[3; 0], [3; 1], [3; 2], [3; 3]\}$ . So

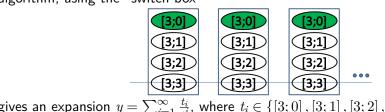
 $\sigma_A(y) = \sum_{i=1}^{\infty} \frac{\sigma_A(t_i)}{A_i}$  as a result of FTC (Prop. 1). Contradiction arises since

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on the one hand  $\langle 3012 \rangle \Rightarrow \text{L.H.S.} = \sigma_A(\partial_2) - \sigma_A(\partial_1) < 0$ ;

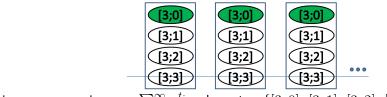
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and  $y = \sum_{i=1}^{\infty} \frac{t_i}{\sigma^i}$ , where  $t_i \in (D-D)$ . If |A| > 1, then  $\sigma_A(y) = \sum_{i=1}^{\infty} \frac{\sigma_A(t_i)}{A^i}$ . **Proposition 2 (Special case**  $\langle 3012 \rangle$ )

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[3;2] [3;2] **[3;2]** 

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[3:3] [3:3]

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Example

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are given by 
$$\frac{\partial_0 \ \partial_1 \ \partial_2 \ \partial_3 \ \partial_4 \ \partial_5}{0 \ 1 \ q/3 \ 2q/3 - 1 \ q - 2 \ q - 1} } \\ 0 \ 1 \ 1.577 \ 2.154 \ 2.732 \ 3.732 \ .$$

It corresponds to the allowed pattern "125034":

#### Proposition 1 (Consequence of FTC) Let A be an algebraic conjugate of $q_i$ and $y \in Y$ . Suppose $\Phi$ satisfies the FTC

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$$\sigma_A(\frac{\partial_1}{\partial_1}) > \sigma_A(\frac{\partial_2}{\partial_2}) > \sigma_A(\frac{\partial_5}{\partial_5}) > \sigma_A(\frac{\partial_0}{\partial_0}) > \sigma_A(\frac{\partial_3}{\partial_3}) > \sigma_A(\frac{\partial_4}{\partial_4})$$

$$\sigma_A(\partial_1) > \sigma_A(\partial_2) > \sigma_A(\partial_5) > \sigma_A(\partial_0) > \sigma_A(\partial_3) > \sigma_A(\partial_4) 
1 > A/3 > A-1 > 0 > 2A/3-1 > A-2 
1 > 0.422 > 0.267 > 0 > -0.154 > -0.732.$$

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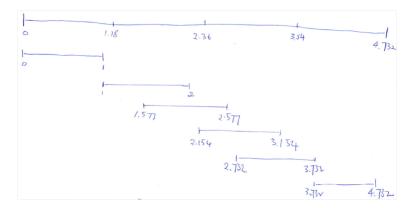
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**Example** m=5,  $q=3+\sqrt{3}\approx 4.732$ , and  $\left\{\partial_i\right\}_{i=0}^m$  with  $0<\partial_{i+1}-\partial_i\leq 1$  are given by  $\partial_0\mid\partial_1\quad\partial_2\quad\partial_3\quad\partial_4\mid\partial_5$ 

0		_	9	-		
0	1	q/3	2q/3 - 1	q-2	q-1	
0	1	1.577	2.154	2.732	3.732	•

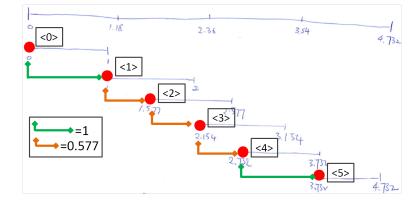
We end by visualizing it. When  $\ell=0$ ,  $\left[q^\ell \mathsf{D}+\cdots+q\mathsf{D}+\mathsf{D}\right]$  gives the following.



**Example** m=5,  $q=3+\sqrt{3}\approx 4.732$ , and  $\{\partial_i\}_{i=0}^m$  with  $0<\partial_{i+1}-\partial_i\leq 1$  are given by  $\partial_0\mid\partial_1\mid\partial_2\mid\partial_2\mid\partial_3\mid\partial_3\mid\partial_3\mid\partial_4\mid\partial_5$ 

00	01	02	0.3	$\circ_4$	0.5	
		/	2q/3 - 1	-		
0	1	1.577	2.154	2.732	3.732	•

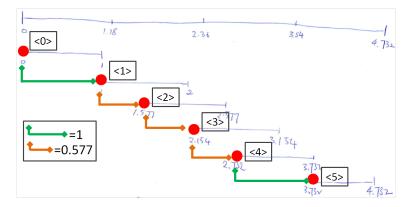
We end by visualizing it. When  $\ell=0$ ,  $\left[q^\ell D+\cdots+qD+D\right]$  gives the following. That is, using the shorthand  $\langle i\rangle:=\partial_i$ , we have



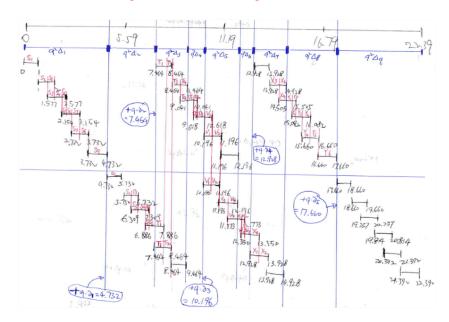
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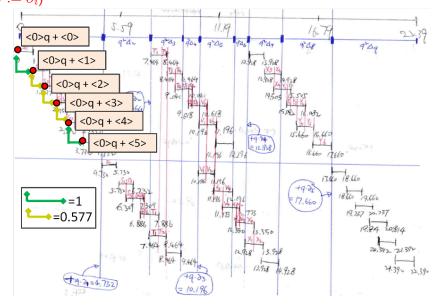
$-2 \mid q-1$	32   3.732
2q/3 - 1  q - 2	2.154 2.732
q/3	1 577
L	
1	7

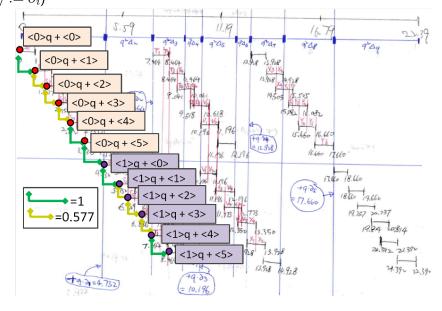
We end by visualizing it. When  $\ell=0$ ,  $\left[q^\ell \mathsf{D}+\cdots+q\mathsf{D}+\mathsf{D}\right]$  gives the following. That is, using the shorthand  $\langle i \rangle:=\partial_i$ , we have

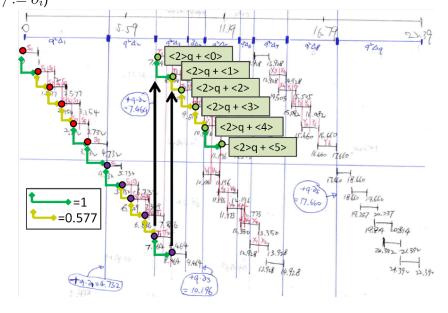


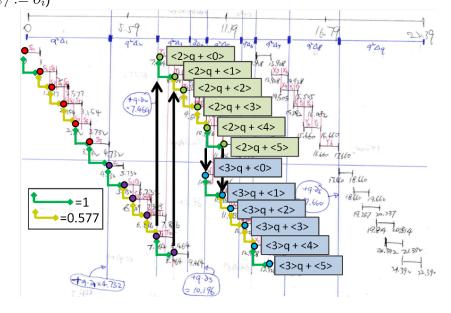
### **Example** When $\ell = 1$ , $\left[ q^{\ell} \mathsf{D} + \dots + q \mathsf{D} + \mathsf{D} \right]$ gives:

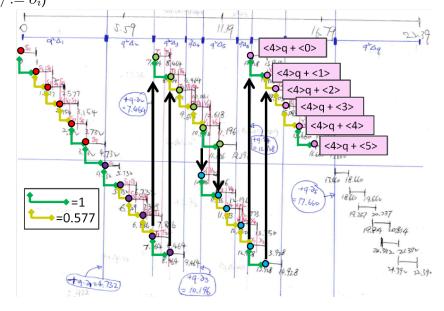


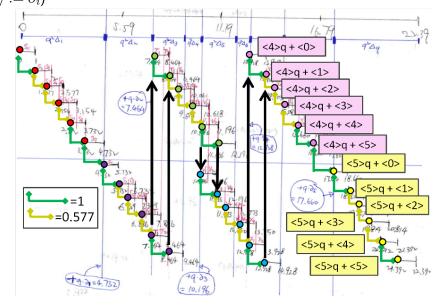


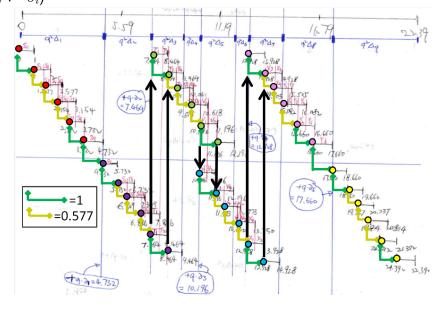


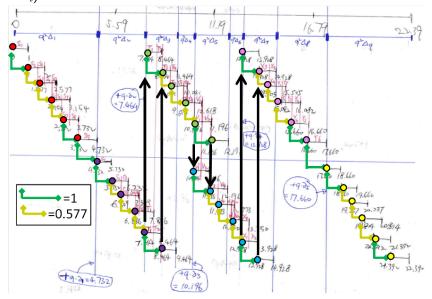




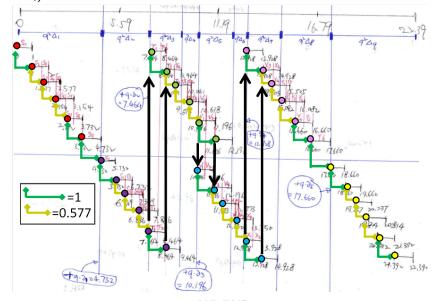








-THE END-



-THE END-