NUMERATION SYSTEMS WITH FINITE TYPE CONDITION

(Part I: Overview)

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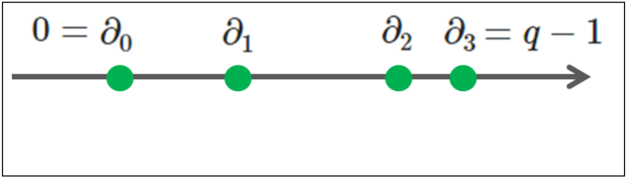

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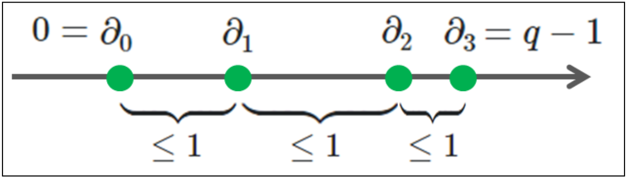

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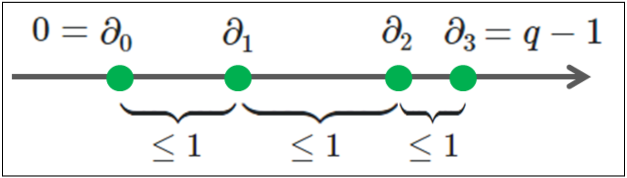

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