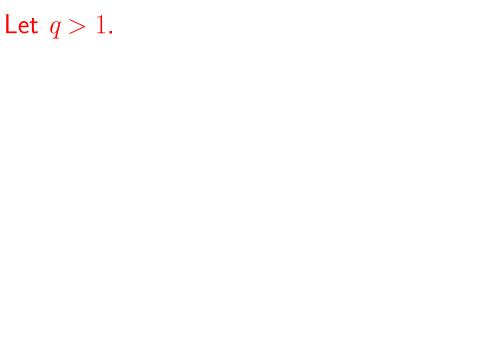
NUMERATION SYSTEMS WITH

FINITE TYPE CONDITION

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(Part I: Overview)

Overview



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$$\mathcal{L}_{0}$$

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, $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$; $\partial_{i+1} - \partial_i \equiv 1$.

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- 3. The special case D = $\left\{\frac{(q-1)0}{m}, \frac{(q-1)1}{m}, \cdots, \frac{(q-1)m}{m}\right\}$ has been well studied (Erdős & Komornik, 1998) (Akiyama & Komornik, 2013).

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Let q > 1 (base), $D := \{\partial_0, \dots, \partial_m\}$, $\partial_i \in \mathbb{R}$ (digits), such that $\partial_0 = 0$, $\partial_m = q - 1$, and $0 < \partial_{i+1} - \partial_i \le 1$ for all i. $\Phi := (q, D)$ is a numeration

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- 3. $[m=4 \text{ and FTC holds}] \Rightarrow |A| < 1 \text{ for all REAL algebraic conjugate}$ A of q.

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1. FTC holds $\Rightarrow q$ is an algebraic integer and $\partial_i \in \mathbb{Q}[q]$ for all i.

2. $[1 \le m \le 3$ and FTC holds] $\Rightarrow |A| < 1$ for all algebraic conjugate A of q. (i.e. q is a PV number)

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4. For each $m \geq 4$, there exists Φ satisfying the FTC, and the associated q is NOT a PV number.

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1. FTC holds $\Rightarrow q$ is an algebraic integer and $\partial_i \in \mathbb{Q}[q]$ for all i.

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